Control Energy Exponents in Large-Scale Networks

Giacomo Baggio

Department of Information Engineering, University of Padova

Focus Period "Network Dynamics and Control"

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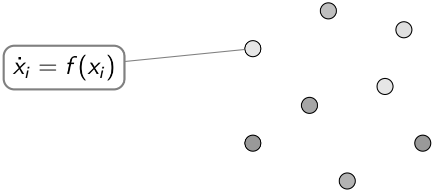
DIPARTIMENTO
DI INGEGNERIA
DELL'INFORMAZIONE

Outline

- Structural vs. practical network controllability
- Control energy scaling and network structure
- Control energy exponents
- Summary and ongoing work

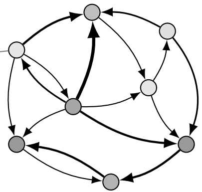
Network controllability, in a nutshell



dynamic units

Network controllability, in a nutshell

$\dot{x}_i = f(x_i, \sum_{j \in \mathcal{N}_i} x_j)$

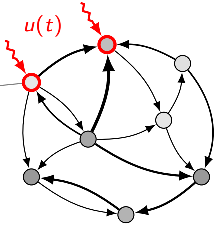


$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

sparsely interacting, dynamic units

Network controllability, in a nutshell

$$\dot{x}_i = f(x_i, \sum_{j \in \mathcal{N}_i} x_j, u_i(t))$$

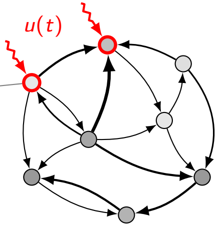


$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

sparsely interacting, dynamic units + sparse actuation

Network controllability, in a nutshell

$$\dot{x}_i = f(x_i, \textstyle\sum_{j \in \mathcal{N}_i} x_j, u_i(t))$$



$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

sparsely interacting, dynamic units + sparse actuation

when and **how easily** can we enforce a desired configuration of $\{x_i\}$?

how this depends on the structure and **size** of $\mathcal{G}$?

(this talk)

A multidisciplinary interest

12 May, 2011

ARTICLE

doi:10.1038/nature10011

Controllability of complex networks

ARTICLE

Received 7 Apr 2015 | Accepted 19 Aug 2015 | Published 1 Oct 2015 DOI: 10.1038/ncomms9414 OPEN

Controllability of structural brain networks

OPEN ACCESS Freely available online PLoS one

Nodal Dynamics, Not Degree Distributions, Determine the Structural Controllability of Complex Networks

T. Bergstrom[3,5]

BRIEF COMMUNICATIONS ARISING

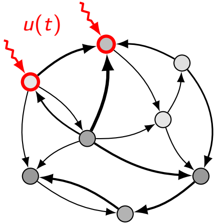
Few inputs can reprogram biological networks

ARISING FROM Y. Liu, J. Slotine & A. Barabási *Nature* **473,** 167–173 (2011)

...

Network controllability: standard setting

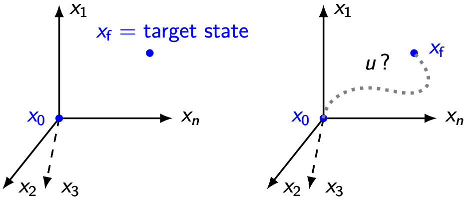
$$\mathcal{G} = (\mathcal{V}, \mathcal{E}), \quad |\mathcal{V}| = n$$



$$\dot{x}(t) = Ax(t) + Bu(t)$$

A = adjacency matrix of $\mathcal{G}$

B selects a subset of control nodes $\mathcal{K} \subseteq \mathcal{V}$



controllability =

$\exists\, u(t), T: \; x(0) = x_0, \; x(T) = x_f, \; \forall x_0, x_f$

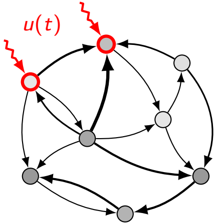
$$\Updownarrow$$

$$\text{rank} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n$$

Kalman rank condition

Network controllability: structural approach

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}), \quad |\mathcal{V}| = n$$



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A = adjacency matrix of $\mathcal{G}$

B selects a subset of control nodes $\mathcal{K} \subseteq \mathcal{V}$

structural controllability =

$\exists$ edge weights s.t. network is controllable

$\Downarrow$

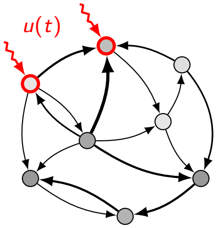
controllability for *almost all* choices of weights!

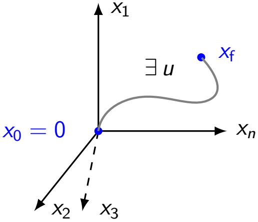
- ✓ captures the role of network topology
- ✓ can be checked via graphical conditions
- ✓ efficient algorithms to find smallest set $\mathcal{K}$ ensuring structural controllability

[Lin, 1974], [Liu et al., 2011],...

Network controllability: practical approach

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}), \quad |\mathcal{V}| = n$$





$$\dot{x}(t) = Ax(t) + Bu(t)$$

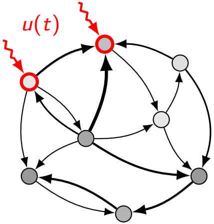
How much energy is needed?

A = adjacency matrix of $\mathcal{G}$

B selects a subset of control nodes $\mathcal{K} \subseteq \mathcal{V}$

Network controllability: practical approach

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}), \quad |\mathcal{V}| = n$$



$$\dot{x}(t) = Ax(t) + Bu(t)$$

A = adjacency matrix of $\mathcal{G}$

B selects a subset of control nodes $\mathcal{K} \subseteq \mathcal{V}$

Controllability Gramian in $[0, T]$:

$$\mathcal{W}_T = \int_0^T e^{At} B B^\top e^{A^\top t} \mathrm{d}t$$

Minimum-energy control input:

$$u^\star(t) = B^\top e^{A^\top (T-t)} \mathcal{W}_T^{-1} x_\mathrm{f}, \quad t \in [0, T]$$

Energy needed to reach x_f:

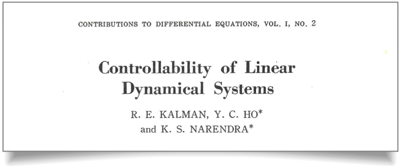
$$\int_0^T \|u^\star(t)\|^2 \mathrm{d}t = x_\mathrm{f}^\top \mathcal{W}_T^{-1} x_\mathrm{f}$$

Network controllability: practical approach

Energy needed to reach x_f:

$$\int_0^T \|u^\star(t)\|^2 dt = x_f^\top \mathcal{W}_T^{-1} x_f$$

[Kalman, Ho, Narendra, 1963]



CONTRIBUTIONS TO DIFFERENTIAL EQUATIONS, VOL. I, NO. 2

Controllability of Linear Dynamical Systems

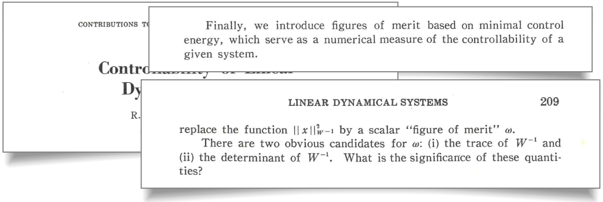
R. E. KALMAN, Y. C. HO*
and K. S. NARENDRA*

Network controllability: practical approach

Energy needed to reach x_f:

$$\int_0^T \|u^\star(t)\|^2 dt = x_f^\top \mathcal{W}_T^{-1} x_f$$

[Kalman, Ho, Narendra, 1963]



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Finally, we introduce figures of merit based on minimal control energy, which serve as a numerical measure of the controllability of a given system.

LINEAR DYNAMICAL SYSTEMS 209

replace the function $\|x\|^2_{W^{-1}}$ by a scalar "figure of merit" ω.
There are two obvious candidates for ω: (i) the trace of W^{-1} and (ii) the determinant of W^{-1}. What is the significance of these quantities?

Control energy metrics:

$\lambda_{\min}^{-1}(\mathcal{W}_T)$ = worst-case control energy for unit norm x_f

$\frac{1}{n}\mathrm{tr}(\mathcal{W}_T^{-1})$ = average control energy for unit norm x_f

(polynomially equivalent in n)

$\sqrt[n]{\det(\mathcal{W}_T^{-1})}$ = volume of reachable set with unit input energy

$\lambda_{\max}(\mathcal{W}_T)$ = best-case control energy for unit norm x_f

$\mathrm{tr}(\mathcal{W}_T) = (\mathcal{H}_2 \text{ system norm})^2$, no direct link to control energy

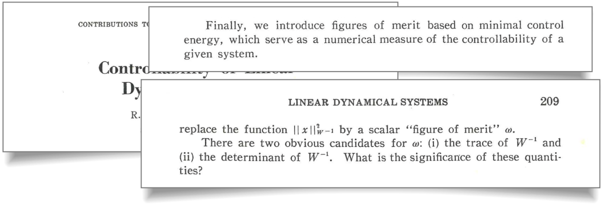
(polynomially equivalent in n)

Network controllability: practical approach

Energy needed to reach x_f:

$$\int_0^T \|u^\star(t)\|^2 dt = x_f^\top \mathcal{W}_T^{-1} x_f$$

[Kalman, Ho, Narendra, 1963]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LINEAR DYNAMICAL SYSTEMS 209

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There are two obvious candidates for ω: (i) the trace of W^{-1} and (ii) the determinant of W^{-1}. What is the significance of these quantities?

For a given control energy metric:

1. How the control energy depends on the structure and size of A?
[Pasqualetti et al., 2014], [Olshevsky, 2016], [Lindmark and Altafini, 2018],...

this talk

2. How to select $\mathcal{K}$ so as to minimize the control energy?
[Summers et al., 2015], [Tzoumas et al., 2016], [Nozari et al., 2019]...

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Difficult- and easy-to-control networks

standing assumption: $\{(A_n, B_n)\}_{n\in\mathbb{N}}$, $A_n \in \mathbb{R}^{n\times n}$ stable, $|\mathcal{K}| \leq m$ indep. of n

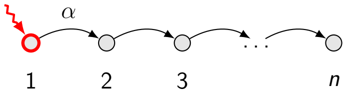
$$\mathcal{W}_n := \int_0^\infty e^{A_n t} B_n B_n^\top e^{A_n^\top t} \mathrm{d}t$$

Def: The growing network with adjacency matrix A_n and input matrix B_n is

- ▷ **difficult-to-control** if $\exists\, k_1, k_2 > 0$ indep. of n s.t. $\lambda_{\min}(\mathcal{W}_n) \leq k_1 e^{-k_2 n}$
- ▷ **easy-to-control** if $\nexists\, k_1, k_2 > 0$ indep. of n s.t. $\lambda_{\min}(\mathcal{W}_n) \leq k_1 e^{-k_2 n}$

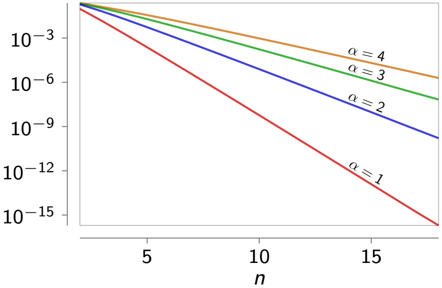
- ▷ difficult-to-control = worst-case control energy grows exponentially or faster in n
- ▷ easy-to-control = worst-case control energy grows less than exponentially in n

Toy example



$$A_n = \begin{bmatrix} -\delta & 0 & \cdots & \cdots & 0 \\ \alpha & -\delta & 0 & \cdots & 0 \\ 0 & \alpha & -\delta & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \alpha & -\delta \end{bmatrix}, \; B_n = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \quad \delta, \alpha > 0$$

$\lambda_{\min}(\mathcal{W}_n) \quad (\delta = 1)$



$\mathcal{W}_n$ linked to symmetric Pascal matrix, closed-form expressions for $\mathcal{W}_n^{-1}$

$$\Downarrow$$

$$\lambda_{\min}(\mathcal{W}_n) \leq k_1 \left(1 + \frac{2\delta}{\alpha}\right)^{-2n} \qquad k_1 > 0 \text{ indep. of } n$$

$$\Downarrow$$

difficult-to-control $\forall\, \alpha, \delta > 0$

A (large) class of difficult-to-control networks

Def: The sequence $\{A_n\}_{n\in\mathbb{N}}$ is **quasi-normal** if A_n is diagonalizable with eigenvector matrix V_n and

$$\exists\, k_1, k_2 > 0 \text{ indep. of } n \;\text{ s.t. }\; \kappa(V_n) = \|V_n\|\|V_n^{-1}\| \le k_1 n^{k_2}$$

$$A_n \text{ quasi-normal} \approx A_n \text{ close to be normal } (A_n A_n^\top = A_n A_n^\top)$$

Theorem: If the following hold:

i) $\{A_n\}$ is quasi-normal

ii) the eigenvalues of A_n belong to a compact set of $\{s \in \mathbb{C},\ \mathrm{Re}(s) < 0\}$

Then, the growing network with adjacency matrix A_n and input matrix B_n is difficult-to-control.

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How difficult is it to control a *difficult-to-control* network?

For difficult-to-control nets with control energy that grows exponentially in n the exponential growth rate can be used as a quantifier of control difficulty

Def: The (worst-case) **control energy exponent** of the growing network described by $\{(A_n, B_n)\}_{n\in\mathbb{N}}$ is

$$\rho_w = \limsup_{n\to\infty} -\frac{1}{n} \ln \lambda_{\min}(\mathcal{W}_n).$$

The larger ρ_w, the more difficult to control the network

How to evaluate ρ_w?

Ingredients

i) $\{A_n\}$ quasi-normal

ii) the eigenvalues of A_n belong to a compact set of $\{s \in \mathbb{C},\ \mathrm{Re}(s) < 0\}$

iii) $\{(A_n, B_n)\}$ robustly controllable

Def:[*] The sequence $\{(A_n, B_n)\}_{n\in\mathbb{N}}$ is **robustly controllable** if

$$\exists\, k_1, k_2 > 0 \text{ indep. of } n \ \text{ s.t. } \ d_{\mathrm{unc}}(A_n, B_n) \geq k_1 \frac{1}{n^{k_2}}$$

where $d_{\mathrm{unc}}(A_n, B_n) = \inf_{\Delta A, \Delta B} \|[\Delta A\ \Delta B]\|$ s.t. $(A + \Delta A, B + \Delta B)$ uncontrollable.

robustly controllable $\implies$ control energy grows *exactly* exponentially in n

[*] [Tsiamis and Pappas, 2022]

Control energy exponents for single-input networks

- $|\mathcal{K}| = 1 \implies \mathcal{W}_n = Q_n \mathcal{C}_n \overline{Q}_n^\top, \quad Q_n = V_n \mathrm{diag}(V_n^{-1} B_n)$

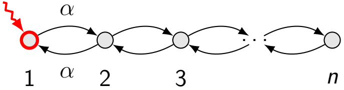
$$[\mathcal{C}_n]_{ij} = -\frac{1}{\lambda_i^{(n)} + \overline{\lambda}_j^{(n)}}, \quad \left\{\lambda_i^{(n)}\right\} \text{ eigenvalues of } A_n$$

- i)-ii) $\implies \lambda_{\min}(\mathcal{W}_n) \approx \lambda_{\min}(\mathcal{C}_n), \quad \mathcal{C}_n$ Cauchy matrix with closed-form $\mathcal{C}_n^{-1}$

Theorem: If $|\mathcal{K}| = 1$, i)-iii) hold and $\{A_n\}$ admits a (sufficiently regular) asymptotic eigenvalue density $\phi(\cdot)$. Then,

$$\rho_w = 2 \max_{\lambda \in \mathrm{supp}(\phi)} \int_{\mathbb{C}} \log \left| \frac{\overline{\lambda} + \mu}{\lambda - \mu} \right| \phi(\mu) \, \mathrm{d}\mu.$$

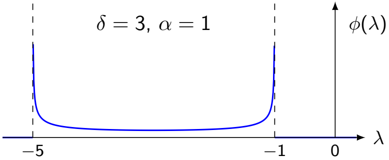
Symmetric line network

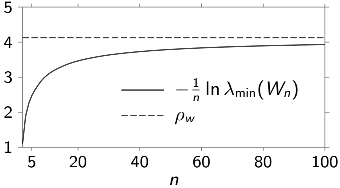


$$A_n = \begin{bmatrix} -\delta & \alpha & \cdots & \cdots & 0 \\ \alpha & -\delta & \alpha & \cdots & 0 \\ 0 & \alpha & -\delta & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \alpha \\ 0 & \cdots & 0 & \alpha & -\delta \end{bmatrix}, \quad B_n = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \quad \delta, \alpha > 0$$

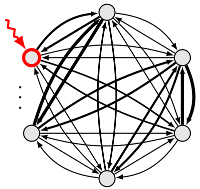
(A_n, B_n) robustly controllable ✓

$$\phi(\lambda) = \frac{\mathbf{1}_{[-\delta-2\alpha, -\delta+2\alpha]}(\lambda)}{\pi\sqrt{4\alpha^2 - (\lambda+\delta)^2}}$$





Complete symmetric random network



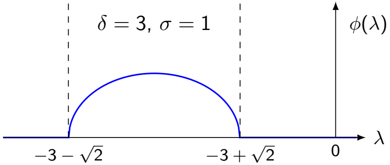
$$A_n = \frac{1}{2\sqrt{n}}(\bar{A}_n + \bar{A}_n^\top) - \delta I, \quad [\bar{A}_n]_{ij} \sim \mathcal{N}(0, \sigma^2), \quad B_n = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \quad \delta > 0$$

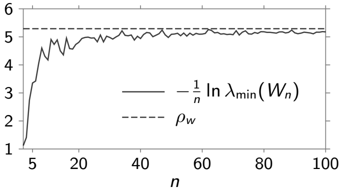
(A_n, B_n) robustly controllable ✓

(asymptotically almost surely)

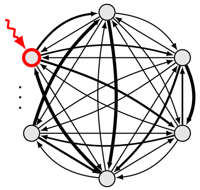
$$\phi(\lambda) = \frac{\sqrt{2\sigma^2 - (\lambda+\delta)^2}}{\pi\sigma^2} \mathbf{1}_{[-\delta-\sqrt{2}\sigma, -\delta+\sqrt{2}\sigma]}(\lambda)$$

(asymptotically almost surely)





Complete random network



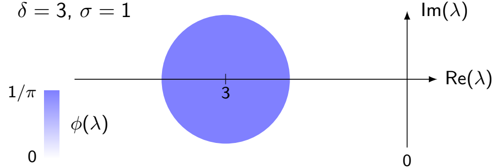
$$A_n = \frac{1}{\sqrt{n}}\bar{A}_n - \delta I, \quad [\bar{A}_n]_{ij} \sim \mathcal{N}(0,\sigma^2), \quad B_n = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \quad \delta > 0$$

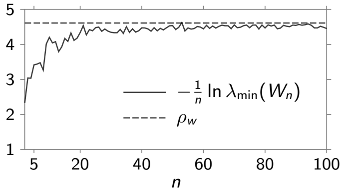
(A_n, B_n) quasi normal (?)

(A_n, B_n) robustly controllable (?)

$$\phi(\lambda) = \frac{\mathbf{1}_{\{|z+\delta|\le\sigma\}}(\lambda)}{\pi\sigma^2}$$

(asymptotically almost surely)





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Take aways

- ▷ Practical controllability examines the energy needed to control a system
- ▷ In (almost) symmetric nets, control energy grows exponentially in the size
- ▷ For single input nets, closed-form expressions for the exponential rate

Ongoing work

- ▷ Theory: control energy exponents in the multi-input case?
- ▷ Applications I: real nets $\rightarrow$ estimates of eigenvalues density?
- ▷ Applications II: quantify (dis)advantage of adding memory to nodes

Thank you

G.B., F. Pasqualetti, S. Zampieri, “Energy-Aware Controllability of Complex Networks”, Annu. Rev. Control Robot. Auton. Syst. 5, 2002

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G.B., S. Zampieri, “Reachable Volume of Large-Scale Linear Network Systems: The Single-Input Case”, CDC 2022

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