Nonlinear dynamics of beliefs over social networks

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Nonlinearity plays a key role in collective behavior and distributed information processing!

Broad areas of interest:

- **Nonlinear dynamics on networks:** collective decision-making, social network dynamics, information and infection spread, dynamics of neurons and neural networks

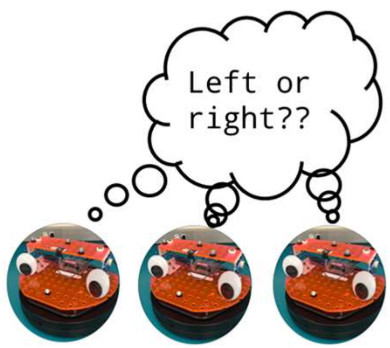
- **Control theory and machine learning for complex systems:** system identification of nonlinear dynamics from data, understanding learning algorithms using control-theoretic tools, reservoir computing

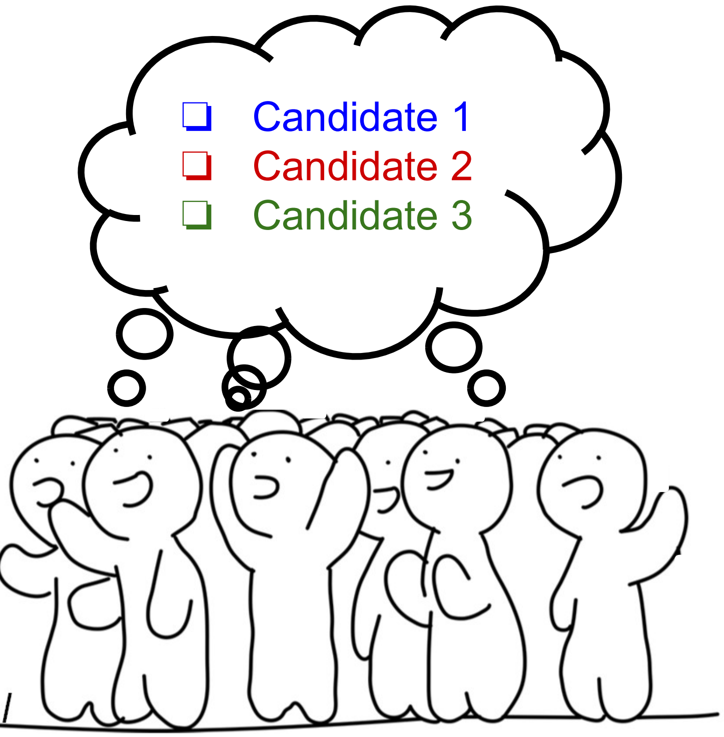
Broad goals of this work

- Develop and analyze **new modeling framework** for multi-alternative belief formation on a social network: understand critical transitions, tunable sensitivity
- Applications to **human social networks**: opinion polarization in online and real-world networks, dynamics of election outcomes, political polarization in governmental bodies, collaborative decision-making
- Applications to **collective animal behavior**: understanding multi-alternative decisions in animal groups, e.g. on spatially embedded options during movement
- Applications to **technological networks**: design of fast and flexible, tunably sensitive collective decisions in autonomous teams, e.g. robotic swarms; bio-inspired algorithm for human-swarm collaboration

Belief formation is a building block of collective behavior

Groups of people, animals, and technological units **evaluate alternatives** to navigate the world, to make decisions, and to solve complex problems together.

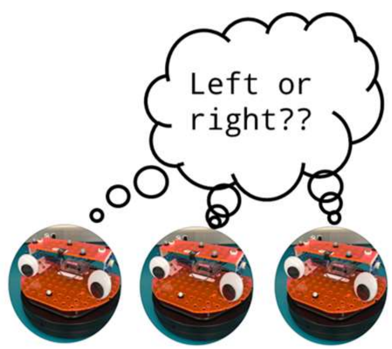


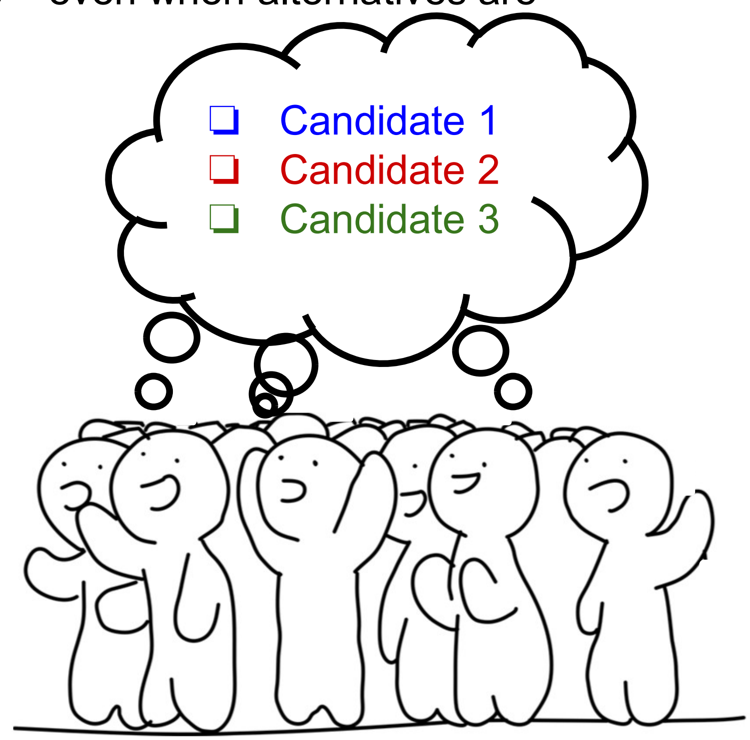


Evaluating alternatives is a **dynamic process** that depends on a balance of **macroscopic** factors *(e.g. social relationships and external information)* and **microscopic** cognitive factors (*e.g. individual biases and cognitive dissonances)*

Belief formation is a building block of collective behavior

Transitions from **global indecision / neutrality** to **global decision / formation of strong beliefs** are characteristic of social systems across contexts – even when alternatives are indistinguishable in value!

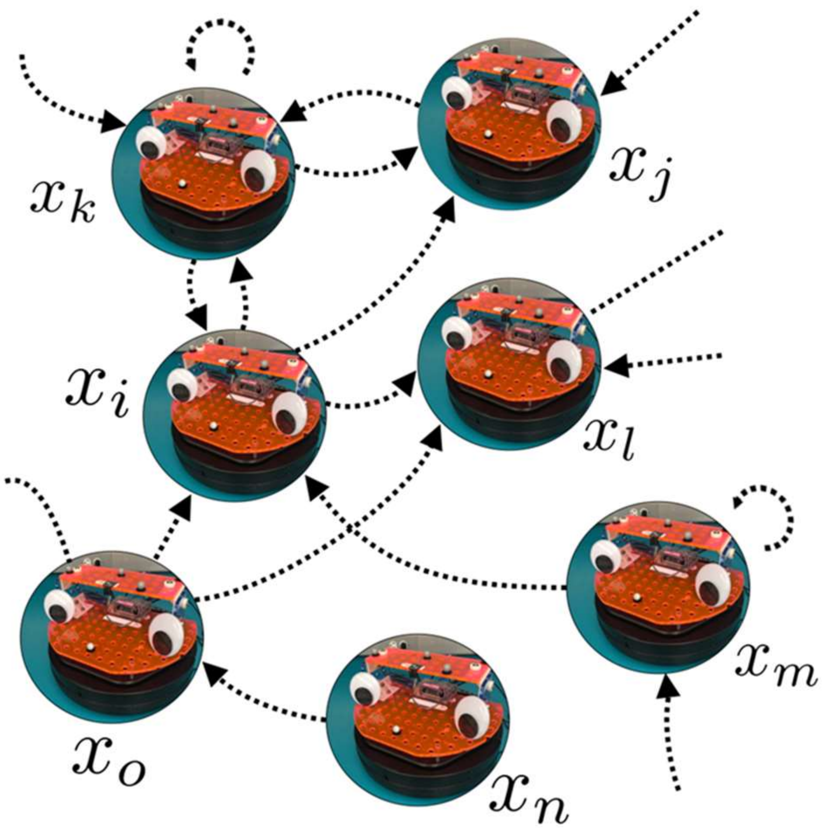




Breaking indecision/deadlock = **bifurcation on a network**

A. Franci, M. Golubitsky, I. Stewart, **A. Bizyaeva**, N.E. Leonard. Breaking indecision in multi-agent, multi-option dynamics, 2023, SIAM Journal on Applied Dynamical Systems

Belief formation: evaluating one option or topic



Agent i can share its belief state $x_i \in \mathbb{R}$ with its neighbors on a communication network

$x_i = 0$ **neutral belief**
$x_i > 0$ **favor**
$x_i < 0$ **disfavor/reject**

Agreement:
$\text{sign}(x_i) = \text{sign}(x_k)$
Disagreement:
$\text{sign}(x_i) \neq \text{sign}(x_k)$

Belief formation via local weighted averaging

Discrete-time averaging:

N agents

$a_{ik} \geq 0$

$$x_i(T+1) = a_{i1}x_1(T) + \cdots + a_{iN}x_N(T)$$

M.H. DeGroot. "Reaching a consensus." *Journal of the American Statistical Association* 69.345 (1974): 118-121.

Continuous-time averaging:

$$\dot{x}_i = \sum_{k=1}^{N} a_{ik}(x_k - x_i) = -d_i x_i + \sum_{\substack{k=1 \\ k \neq i}}^{N} a_{ik} x_k$$

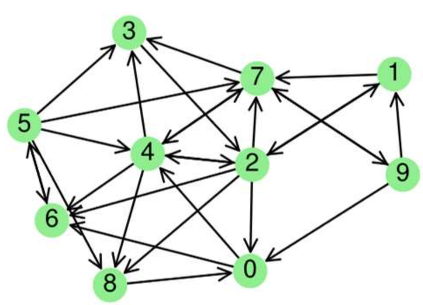
R. Olfati-Saber and R. M. Murray. "Consensus problems in networks of agents with switching topology and time-delays." *IEEE Transactions on Automatic Control* 49.9 (2004): 1520-1533.

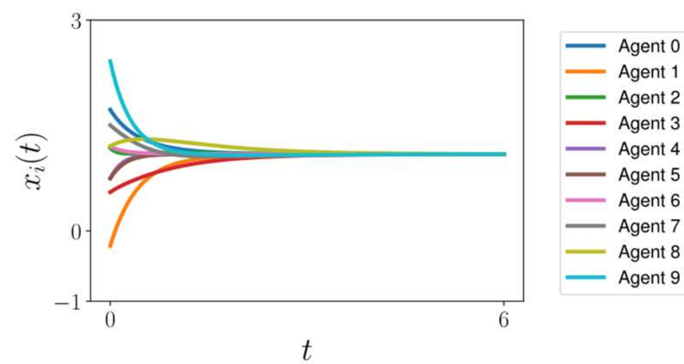
Belief formation via local weighted averaging

$$\dot{x}_i = \sum_{k=1}^{N} a_{ik}(x_k - x_i)$$

Paradox in linear opinion formation models: agents' influence on one another scales with their opinion difference! (W. Mei et.al., Physical Review Research, 2022)

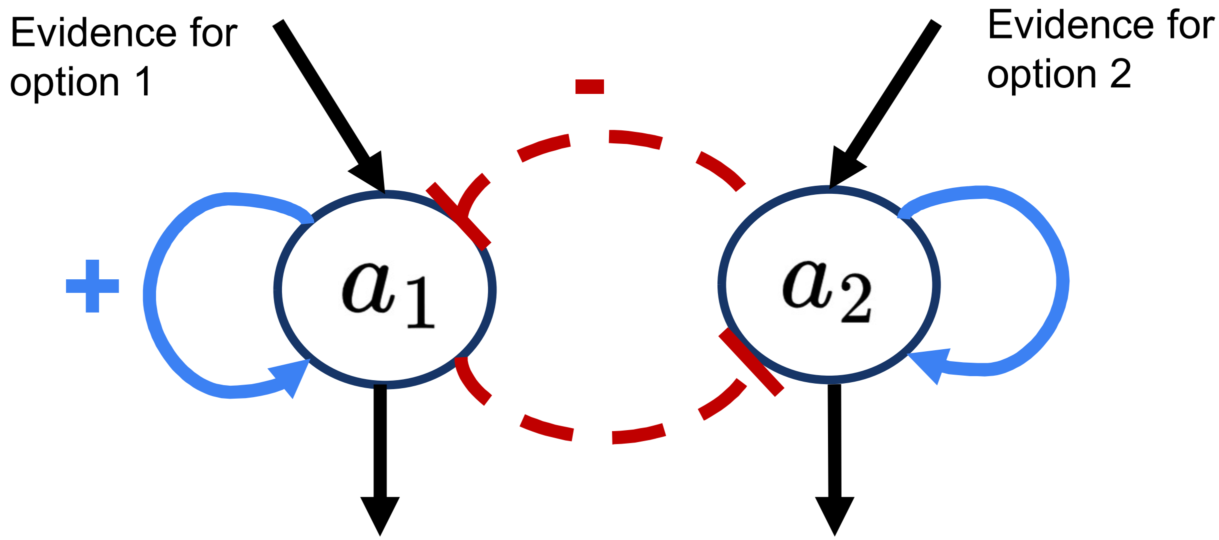
A linear averaging process on any strongly connected network will always reach consensus.





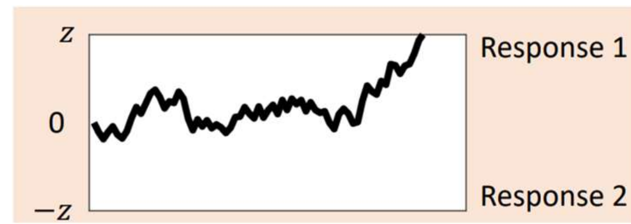
Beyond averaging: nonlinearity in cognitive processing

Multi-alternative decisions result from a **dynamic, nonlinear** evidence accumulation process in the brain

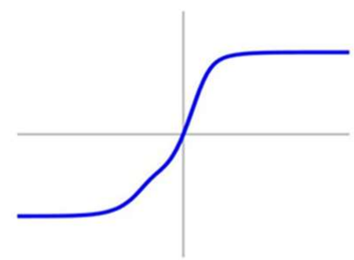


$$\dot{a}_i = -a_i + S(g(a_i - a_j + I_i))$$

Usher & McClelland. The time course of perceptual choice: the leaky, competing accumulator model. *Psychological Review*, 2001



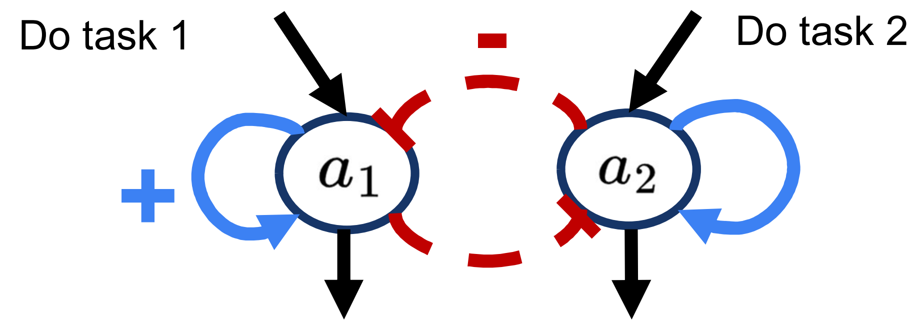
$$z = a_1 - a_2$$

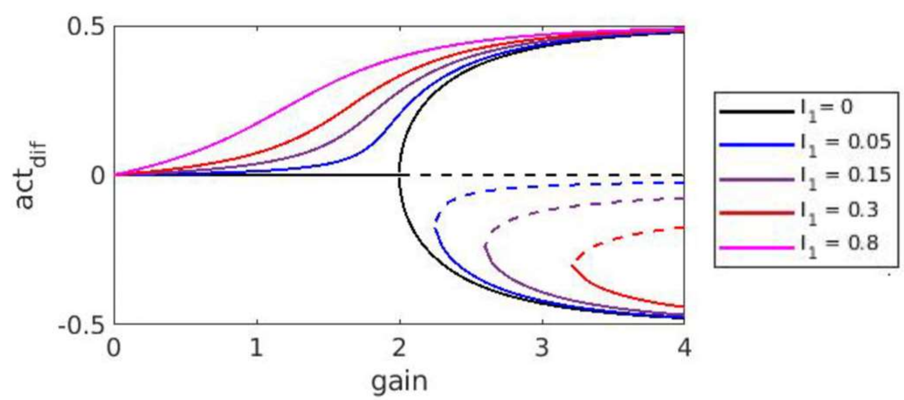


Nonlinear processing in human decision-making

Stability-flexibility dilemma: switching between decisions has cognitive cost! More focused = more costly to switch

$$\dot{a}_i = -a_i + S(g(a_i - a_j + I_i))$$

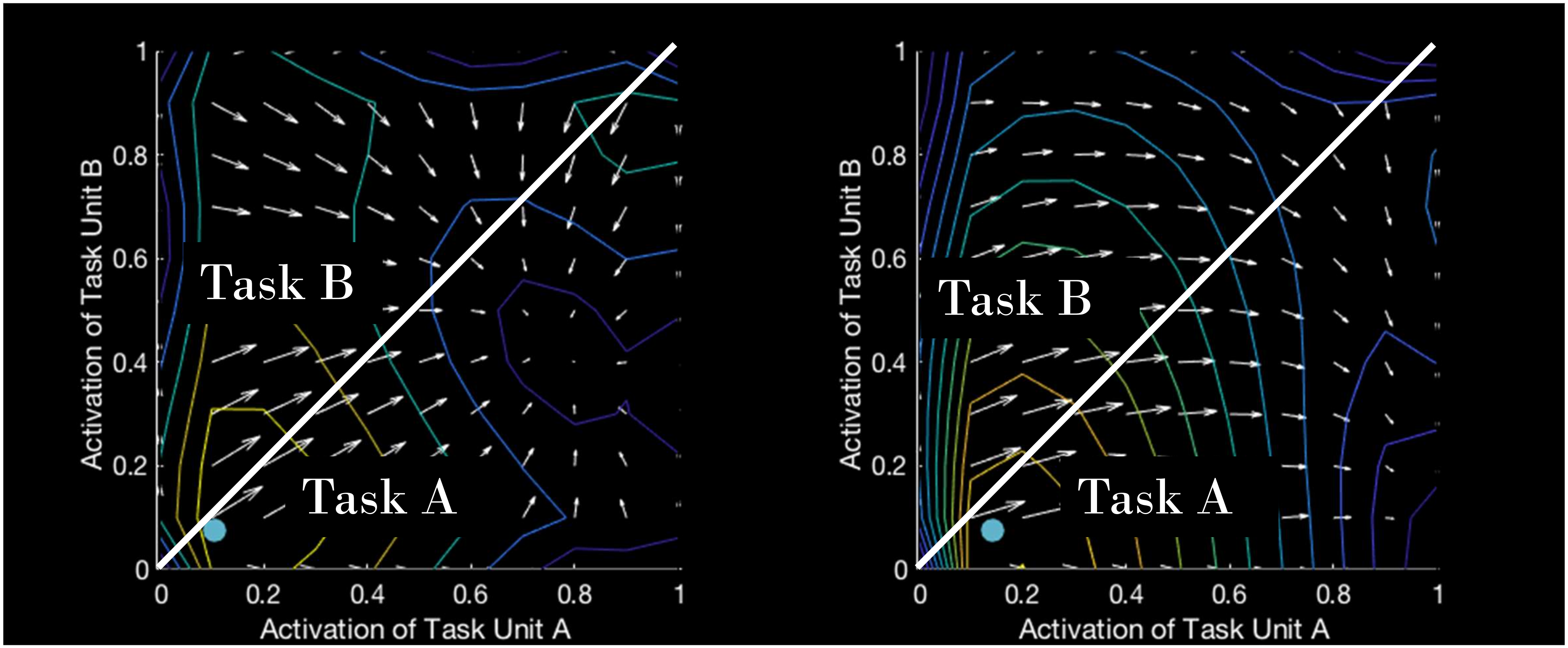




Musslick, **Bizyaeva,** Agaron, Leonard, Cohen. Stability-flexibility dilemma in cognitive control: a dynamical system perspective. *Proc. Cog. Sci, 2019*

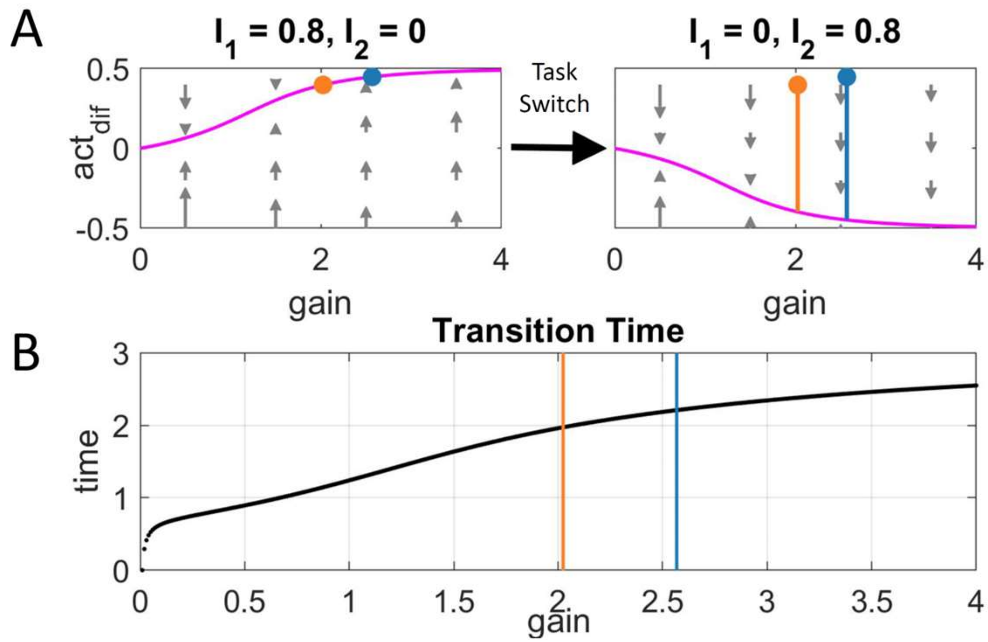
Low gain

High gain



Simulation by Sebastian Musslick

Nonlinear processing in human decision-making

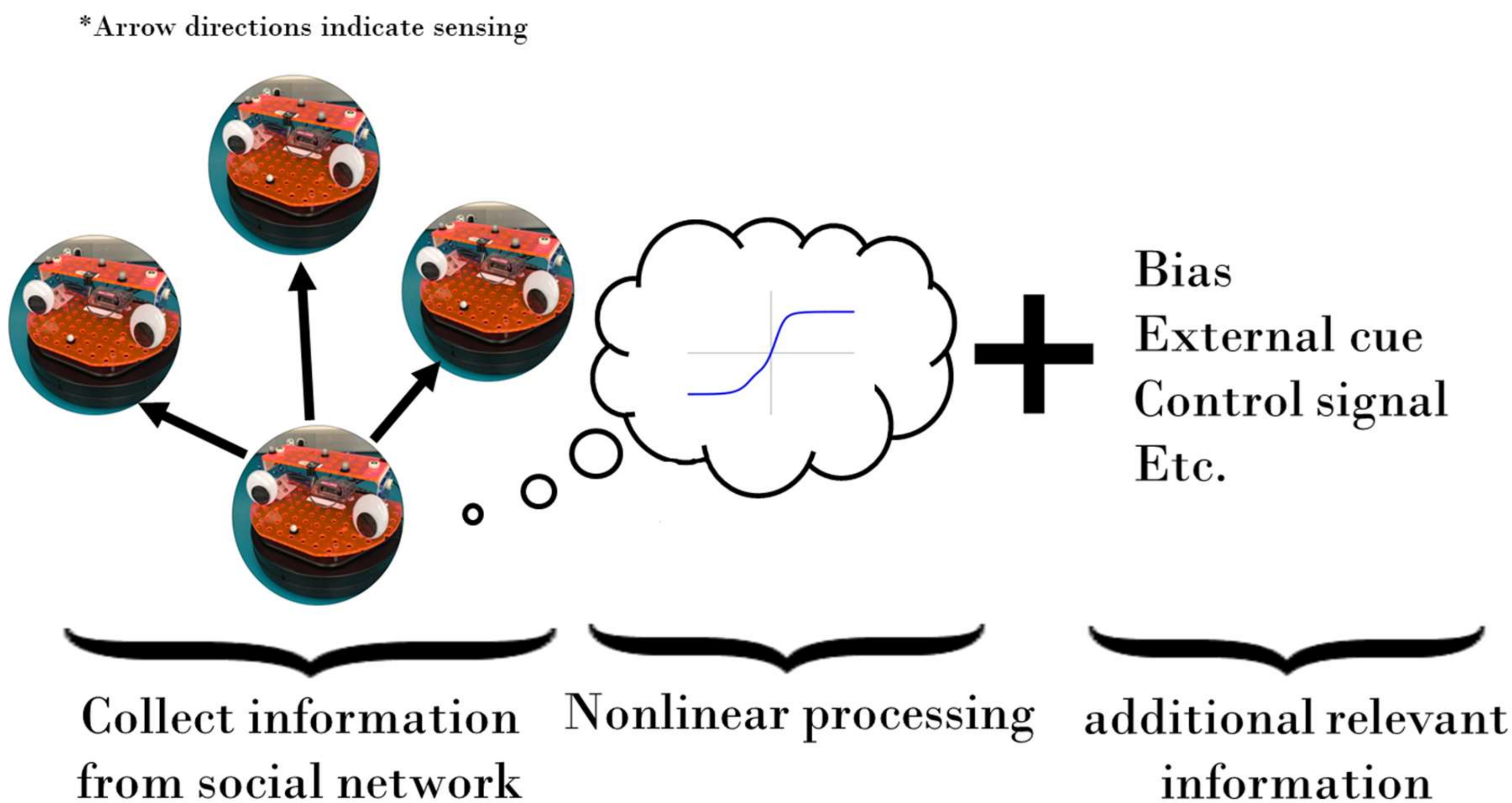


High task switch group

Low task switch group

Musslick, **Bizyaeva,** Agaron, Leonard, Cohen. Stability-flexibility dilemma in cognitive control: a dynamical system perspective. *Proc. Cog. Sci, 2019*

Nonlinear belief formation model on a social network



Nonlinear belief formation model on a social network

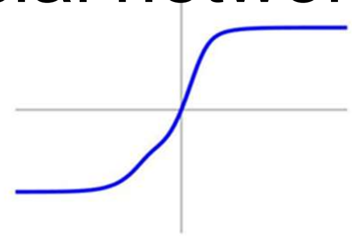
Attention to social interactions

External input or intrinsic bias

$$\dot{x}_i = -d_i x_i + u_i S\left(\alpha x_i + \gamma \sum_{\substack{k=1 \\ k \neq i}}^{N_a} A_{ik}^a x_k\right) + b_i$$

Resistance to forming strong beliefs

Social imitation and self-excitation

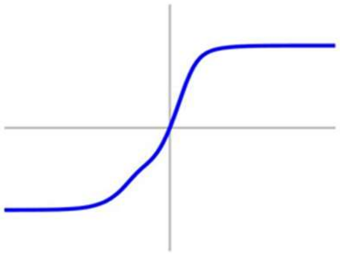


$$S(0) = 0$$
$$S'(0) = 1$$
$$d_i > 0$$
$$u_i \geq 0$$
$$b_i \in \mathbb{R}$$

[1] **A. Bizyaeva**, A. Franci, N.E. Leonard. Nonlinear opinion dynamics with tunable sensitivity, *IEEE Transactions on Automatic Control*, 2023
[2] A. Franci, M. Golubitsky, I. Stewart, **A. Bizyaeva**, N.E. Leonard. Breaking indecision in multi-agent, multi-option dynamics, SIAM Journal on Applied Dynamical Systems, 2023
[3] **A. Bizyaeva**, A. Franci, N.E. Leonard. Bifurcations in nonlinear multi-topic belief formation networks, *arXiv:2308.02755 [physics.soc-ph]*

Nonlinear belief formation model on a social network

$$\dot{x}_i = -d_i x_i + u_i S\left(\alpha x_i + \gamma \sum_{\substack{k=1 \\ k \neq i}}^{N_a} A^a_{ik} x_k\right) + b_i$$



$S(0) = 0$

$S'(0) = 1$

$d_i > 0$

$u_i \geq 0$

$b_i \in \mathbb{R}$

N_a **agents;**

Social parameters:

$\alpha \geq 0$ **strength of self-reinforcement**

$\gamma \geq 0$ **strength of social imitation**

$A^a_{ik} \in \{0, 1, -1\}$ **social relationships (cooperative or antagonistic)**

$A^a = (A^a_{ik})$ **adjacency matrix of signed communication graph** $\mathcal{G}^a$

Beyond scalar beliefs

Emerging perspective: beliefs are *“embedded in a multidimensional, self-sustaining system of mental representations and shaped and reinforced continuously in the social interactions people have in their communities”*

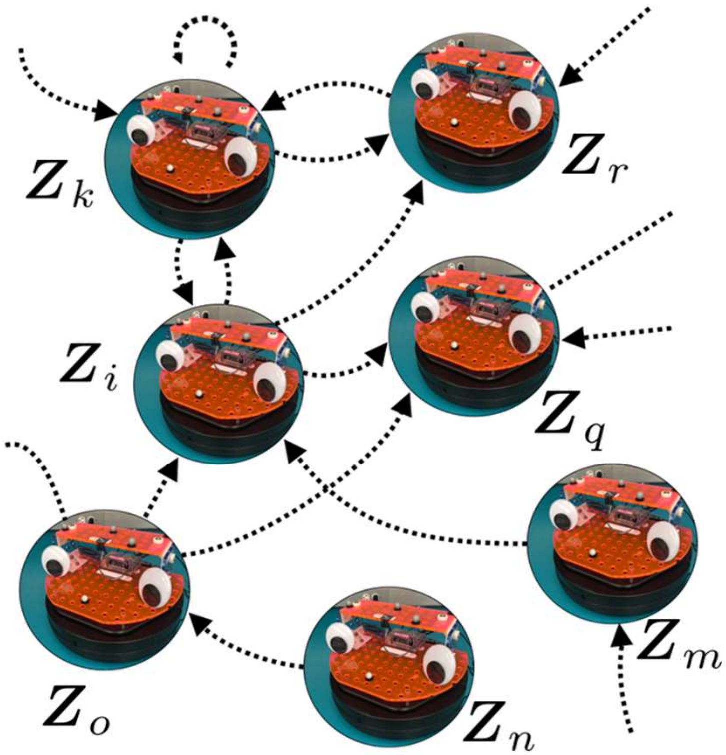
Vlasceanu, M., Dyckovsky, A. M., & Coman, A. (2023). A network approach to investigate the dynamics of individual and collective beliefs. Perspectives on Psychological Science

We need to consider networked relationships not only **between** beliefs of individuals, but also **within** the cognition of each individual!

There is an overarching **belief system** that governs logical relationships between various views of an individual – e.g. left-right ideological spectrum, logical constraints. This must be accounted for explicitly in mathematical models.

Converse, Philip E. "The nature of belief systems in mass publics (1964)” Critical Review

Beyond scalar beliefs: multiple options



N_a **agents,** N_o **options or topics**

$z_{ij} \in \mathbb{R}$ **opinion or belief of agent** i **on topic** j

$$z_{ij} = 0 \quad \text{neutral belief}$$
$$z_{ij} > 0 \quad \text{favor topic } j$$
$$z_{ij} < 0 \quad \text{disfavor topic } j$$

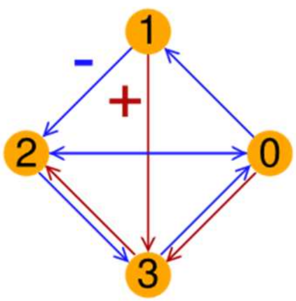
$\boldsymbol{Z}_i = (z_{i1}, \ldots, z_{iN_o})$ **belief state vector for agent** i

Agreement on topic j**:**

$$\text{sign}(z_{ij}) = \text{sign}(z_{kj})$$

Two graphs

Communication graph $\mathcal{G}^a$

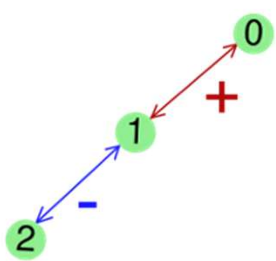


$A^a_{ik} = 0$ agent i does not see agent k

$A^a_{ik} = 1$ agent i is cooperative towards agent k

$A^a_{ik} = -1$ agent i is antagonistic towards agent k

Belief system graph $\mathcal{G}^o$



$A^o_{jl} = 0$ topic j is independent of topic l

$A^o_{jl} = 1$ topic j is positively aligned with topic l

$A^o_{jl} = -1$ topic j is negatively aligned with topic l

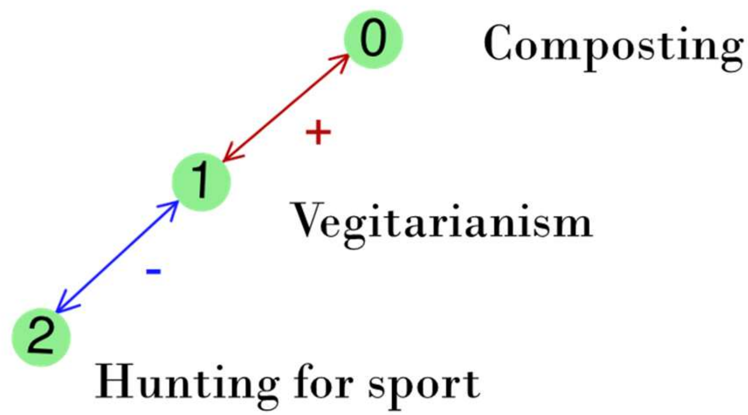
Two graphs

Belief system graph $\mathcal{G}^o$

$A^o_{jl} = 0$ topic j is independent of topic l

$A^o_{jl} = 1$ topic j is positively aligned with topic l

$A^o_{jl} = -1$ topic j is negatively aligned with topic l



A **belief system** encodes the logical, psychological, or social constraints on the relationships between beliefs on different alternatives (Converse, 1964)

Networks of Beliefs theory

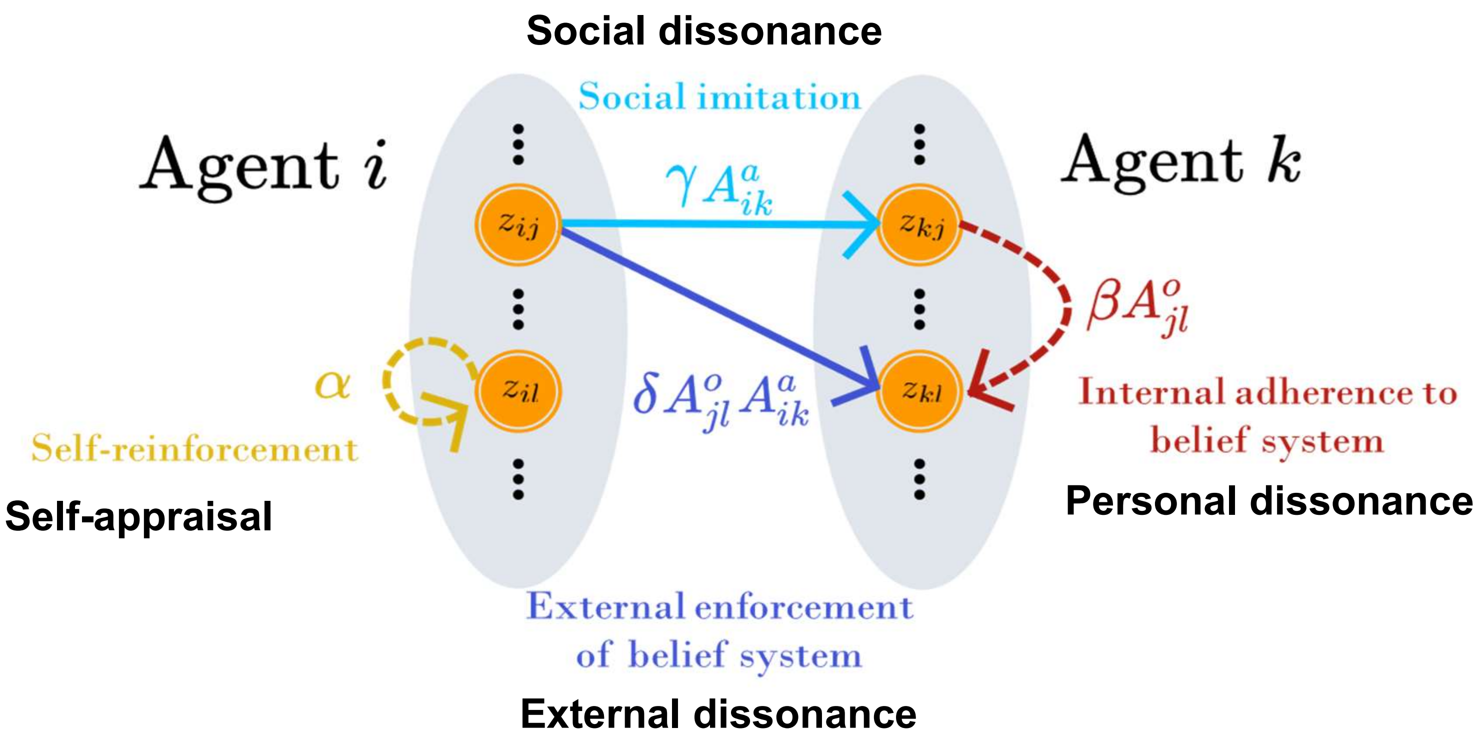
Main premise: belief formation and belief change is the outcome of individuals trying to minimize **cognitive dissonance** from several distinct sources:

- **Personal dissonance,** e.g. I'm vegetarian but I like hunting
- **Social dissonance,** e.g. I'm vegetarian but my friends are not
- **External dissonance,** e.g. I'm vegetarian but my friends are hunters

J. Dalege, M. Galesic, and H. Olsson, "Networks of beliefs: an integrative theory of individual-and social-level belief dynamics," 2023.

Four distinct effects

Vector of individual belief representations



Parameters represent relative levels of attention individuals allocate towards different sources of cognitive dissonance

Belief formation model: multiple options

Attention/urgency/susceptibility to social influence

External input or intrinsic bias

$$\frac{dz_{ij}}{dt} = -d\, z_{ij} + u S_1\left(\alpha z_{ij} + \gamma \sum_{\substack{k=1\\k\neq i}}^{N_a} A^a_{ik} z_{kj}\right) + u \sum_{\substack{l=1\\l\neq j}}^{N_o} S_2\left(\beta A^o_{jl} z_{il} + \delta A^o_{jl} \sum_{\substack{k=1\\k\neq i}}^{N_a} A^a_{ik} z_{kl}\right) + b_{ij}$$

Resistance to forming strong beliefs

Effect of social imitation

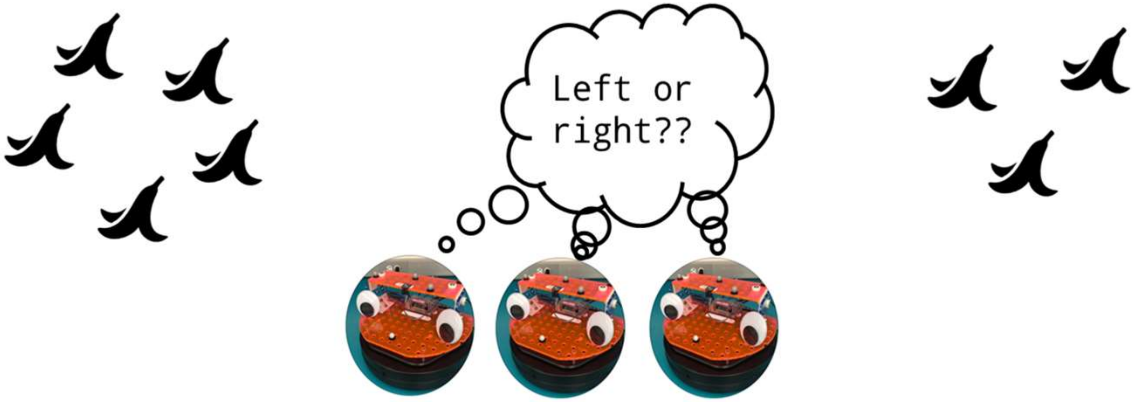
Effect of belief system or logical relationships between alternatives

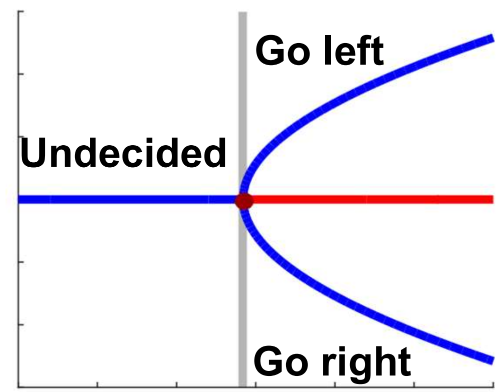
What can we say about this model analytically?

$b_{ij} = 0 \implies \mathbf{Z} = 0$ (indecision) is always an equilibrium!

[1] **A. Bizyaeva**, A. Franci, N.E. Leonard. Nonlinear opinion dynamics with tunable sensitivity, *IEEE Transactions on Automatic Control*, 2023
[2] A. Franci, M. Golubitsky, I. Stewart, **A. Bizyaeva**, N.E. Leonard. Breaking indecision in multi-agent, multi-option dynamics, arXiv:2206.14893, 2022; in press in SIAM Journal on Applied Dynamical Systems
[3] **A. Bizyaeva**, A. Franci, N.E. Leonard. Bifurcations in nonlinear multi-topic belief formation networks, *arXiv:2308.02755 [physics.soc-ph]*

Breaking indecision → bifurcation!





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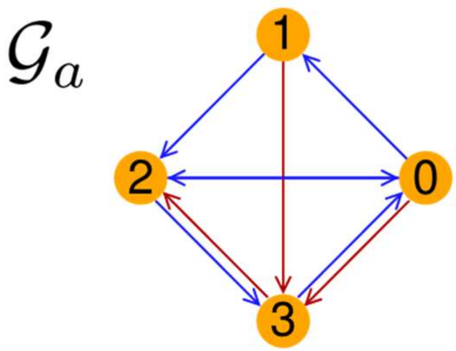
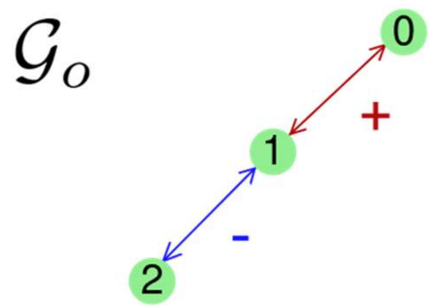
Numerical example: four agents three options

$$\dot{z}_{ij} = -d\, z_{ij} + u S_1\left(\alpha z_{ij} + \gamma \sum_{\substack{k=1 \\ k\neq i}}^{N_a} A^a_{ik} z_{kj}\right) + u \sum_{\substack{l=1 \\ l\neq j}}^{N_o} S_2\left(\beta A^o_{jl} z_{il} + \delta A^o_{jl} \sum_{\substack{k=1 \\ k\neq i}}^{N_a} A^a_{ik} z_{kl}\right)$$





$$A_a = \begin{pmatrix} 0 & -1 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

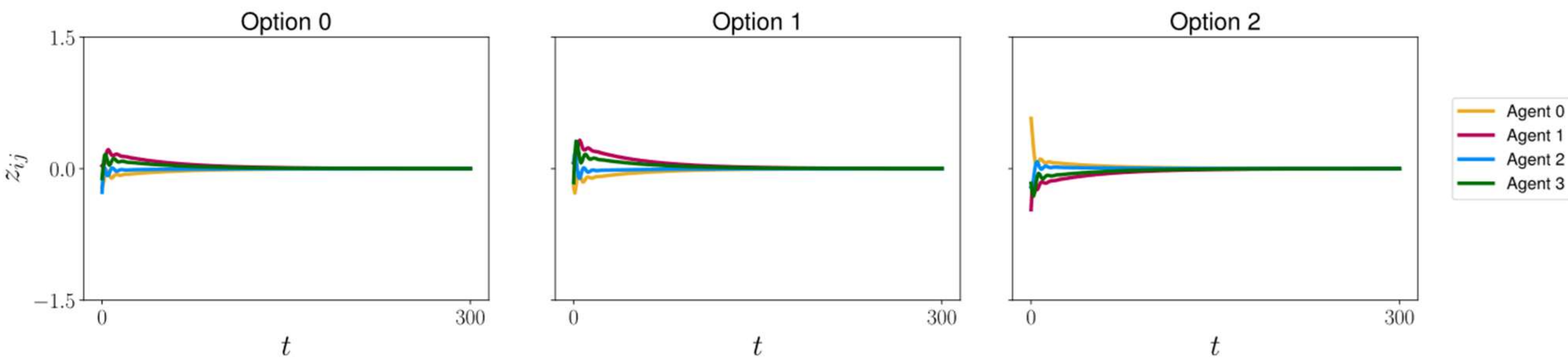
$$A_o = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

Numerical example: four agents three options

$$\dot{z}_{ij} = -d\, z_{ij} + u S_1\left(\alpha z_{ij} + \gamma \sum_{\substack{k=1 \\ k\neq i}}^{N_a} A^a_{ik} z_{kj}\right) + u \sum_{\substack{l=1 \\ l\neq j}}^{N_o} S_2\left(\beta A^o_{jl} z_{il} + \delta A^o_{jl} \sum_{\substack{k=1 \\ k\neq i}}^{N_a} A^a_{ik} z_{kl}\right)$$

$$S_1(\cdot) = \tanh(\cdot) \qquad S_2(\cdot) = \frac{1}{2}\tanh(2\ \cdot)$$

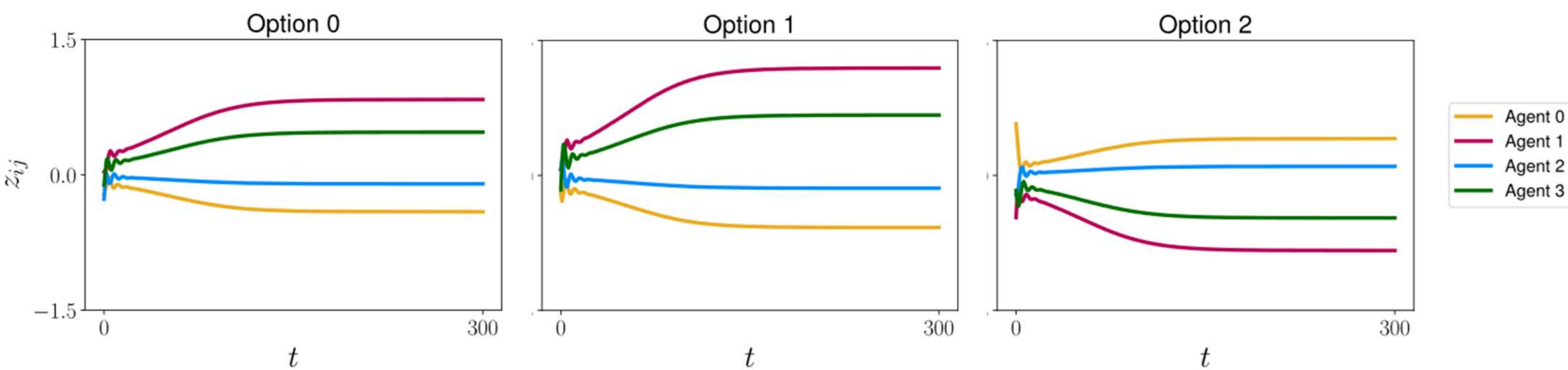
$$d = 1,\ \alpha = \beta = \gamma = \delta = 0.1,\ u = 2.43$$



Numerical example: four agents, three options

$$\dot{z}_{ij} = -d\, z_{ij} + u S_1\left(\alpha z_{ij} + \gamma \sum_{\substack{k=1 \\ k\neq i}}^{N_a} A^a_{ik} z_{kj}\right) + u \sum_{\substack{l=1 \\ l\neq j}}^{N_o} S_2\left(\beta A^o_{jl} z_{il} + \delta A^o_{jl} \sum_{\substack{k=1 \\ k\neq i}}^{N_a} A^a_{ik} z_{kl}\right)$$

$$d = 1,\ \alpha = \beta = \gamma = \delta = 0.1,\ u = 2.53$$



Linearization about indecision equilibrium

$$J(\mathbf{0}) = (-d + u\alpha)\mathcal{I}_{N_a} \otimes \mathcal{I}_{N_o} + u\gamma A_a \otimes \mathcal{I}_{N_o} + u\beta \mathcal{I}_{N_a} \otimes A_o + u\delta A_a \otimes A_o$$

$\sigma(A)$ **: spectrum of** A

Proposition (Eigenvalues and eigenvectors)

1) For each $\eta \in \sigma\big(J(\mathbf{0})\big)$, *there exists* $\lambda \in \sigma(A_a)$ *and* $\mu \in \sigma(A_o)$ *so that*

$$\eta = -d + u(\alpha + \gamma\lambda + \beta\mu + \delta\lambda\mu) := \eta(u, \lambda, \mu);$$

2) Suppose λ_i *is an eigenvalue of* A_a *with a right (left) eigenvector* $\mathbf{v}_{a,i} \in \mathbb{R}^{N_a}$ *and* μ_j *is an eigenvalue of* A_o *with a right (left) eigenvector* $\mathbf{v}_{o,j} \in \mathbb{R}^{N_o}$, *then the vector*

$$\mathbf{v}_{ij} = \mathbf{v}_{a,i} \otimes \mathbf{v}_{o,j} \in \mathbb{R}^{N_a N_o}$$

is a right (left) eigenvector of $J(\mathbf{0})$ *with corresponding eigenvalue* $\eta(u, \lambda_i, \mu_j)$.

$$\eta(u,\lambda,\mu) = -d + u(\alpha + \gamma\lambda + \beta\mu + \delta\lambda\mu)$$
$$\lambda \in \sigma(A_a) \qquad \mu \in \sigma(A_o)$$

Indecision-breaking bifurcation

Proposition (Attention threshold)

Suppose $\mathrm{Re}(\alpha + \gamma\lambda + \beta\mu + \delta\lambda\mu) > 0$ *for any pair* $(\lambda,\mu) \in \Lambda$. *Then there exists a critical value of attention*

$$u^* = \frac{d}{\mathrm{Re}(\alpha + \gamma\lambda + \beta\mu + \delta\lambda\mu)} > 0.$$

The network indecision equilibrium $\boldsymbol{Z} = \mathbf{0}$ *is locally exponentially stable for all* $u < u^*$ *and unstable for all* $u > u^*$.

Let $k = |\Lambda|$. *There exists a* $(k+1)$*-dimensional invariant center manifold* $W^c \subset \mathbb{R}^{N_a N_o + 1}$ *passing through* $(\boldsymbol{Z}, u) = (\mathbf{0}, u^*)$, *tangent to the null space* $\mathcal{N}(J(\mathbf{0}))$ *at* $u = u^*$. *All trajectories of the dynamics starting at* $(\boldsymbol{Z}, u)$ *near* $(\mathbf{0}, u^*)$ *converge to* W^c *exponentially as* $t \to \infty$.

Indecision-breaking bifurcation

$$J(\mathbf{0}) = (-d + u\alpha)\mathcal{I}_{N_a} \otimes \mathcal{I}_{N_o} + u\gamma A_a \otimes \mathcal{I}_{N_o} + u\beta \mathcal{I}_{N_a} \otimes A_o + u\delta A_a \otimes A_o$$

Bifurcation threshold in attention parameter $u^* = \dfrac{d}{\alpha + \gamma\,\mathrm{Re}(\lambda) + \beta\,\mathrm{Re}(\mu) + \delta\,\mathrm{Re}(\lambda\mu)}$ $\lambda \in \sigma(A_a)$, $\mu \in \sigma(A_o)$

How many Jacobian eigenvalues cross the imaginary axis at this singular point?
What is the structure of the center manifold for the local bifurcation?

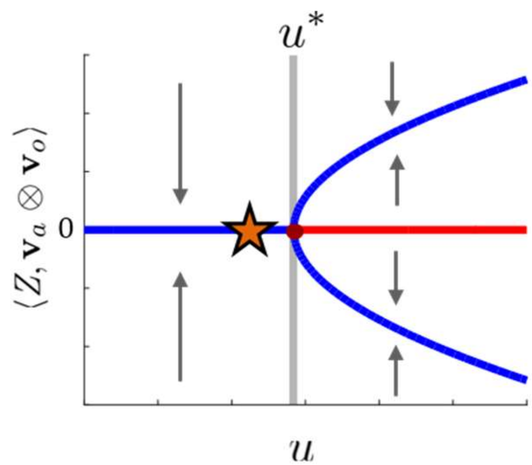
→ Depends on $\mathcal{G}_a$, $\mathcal{G}_o$

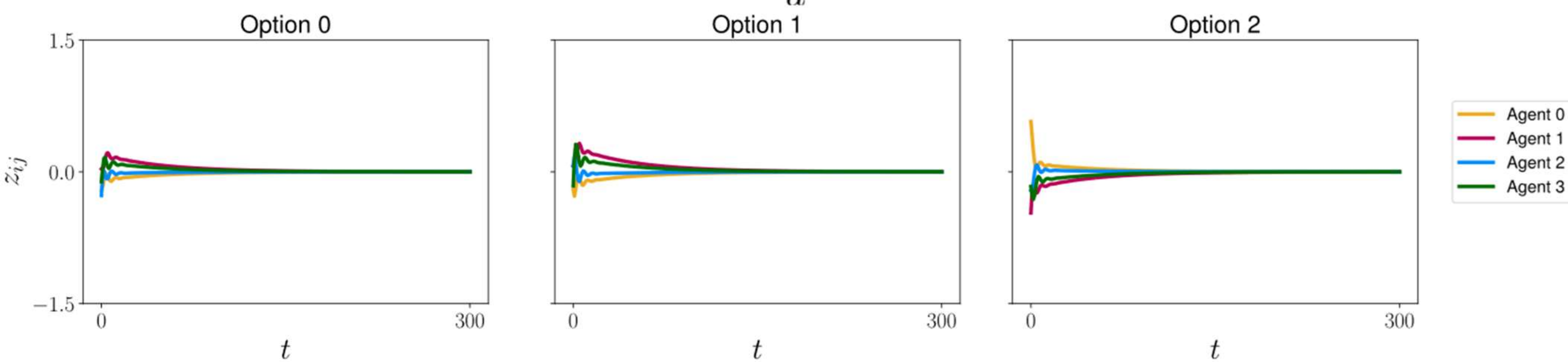
$$(\lambda, \mu) = \mathrm{argmax}_{\lambda_i \in \sigma(A_a), \mu_j \in \sigma(A_o)}\, \gamma\,\mathrm{Re}(\lambda_i) + \beta\,\mathrm{Re}(\mu_j) + \delta\,\mathrm{Re}(\lambda_i \mu_j)$$

Simplest case: (λ, μ) real, simple → pitchfork bifurcation along span of eigenvector $\mathbf{v}_a \otimes \mathbf{v}_o$

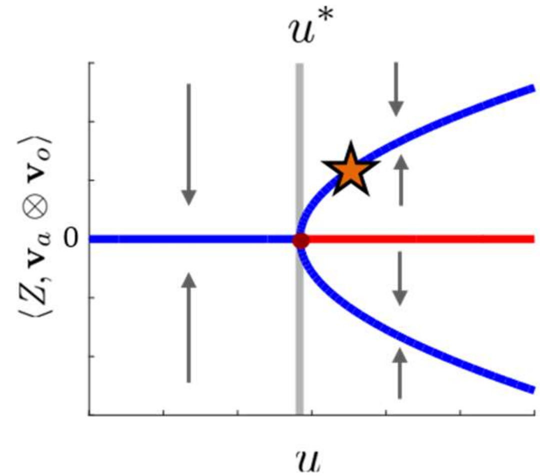
Lyapunov-Schmidt reduction used to classify bifurcation type, local stability of solution branches

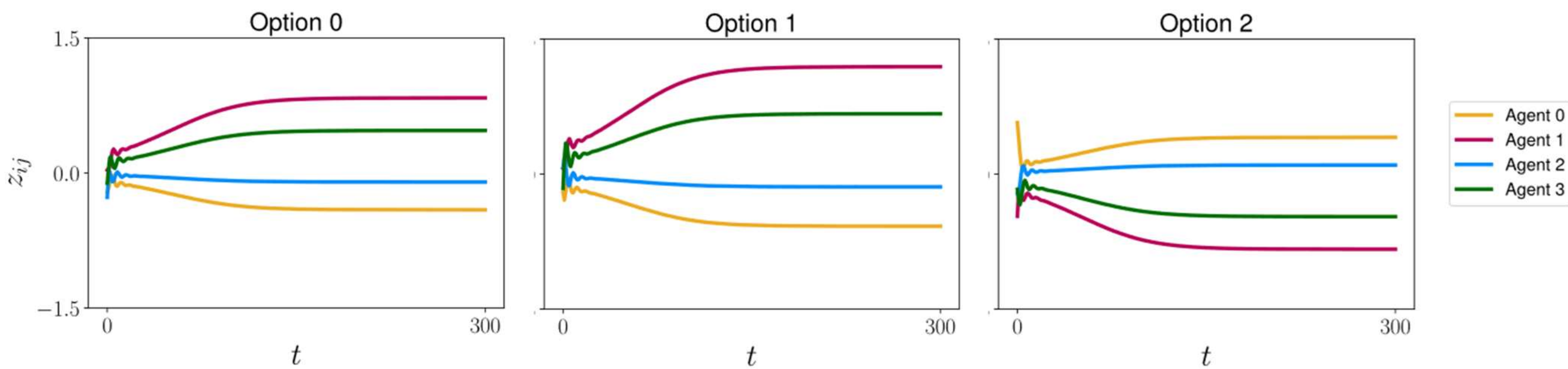
Pitchfork bifurcation





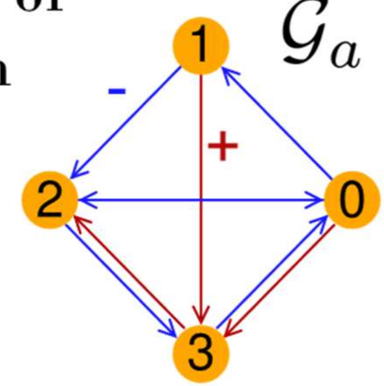
Pitchfork bifurcation



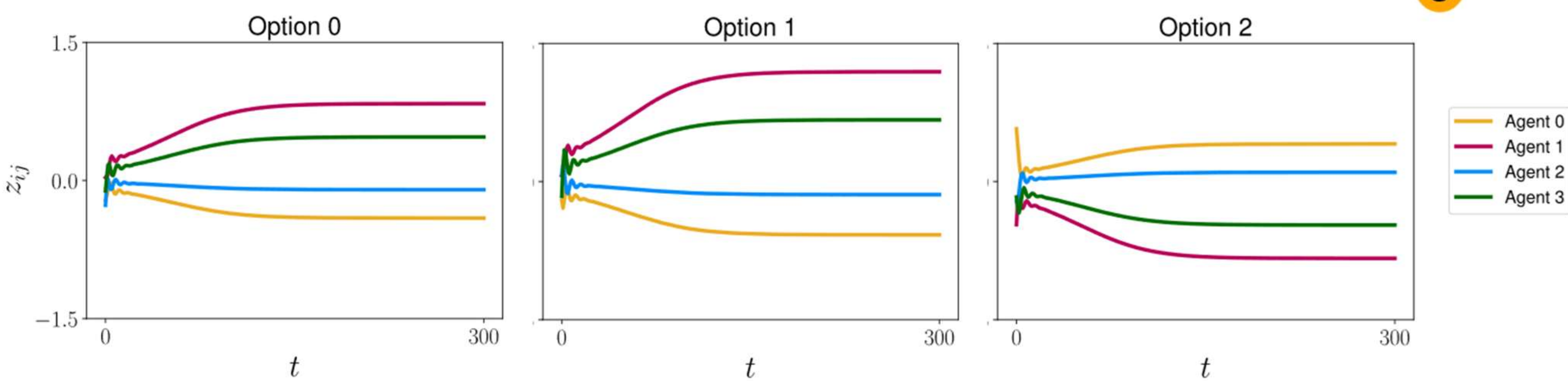


Role of graph structure: social network

Bifurcation happens along $\text{span}(\mathbf{v}_a \otimes \mathbf{v}_o)$: eigenvector of **communication graph** **organizes relationships between agents' beliefs about each topic**

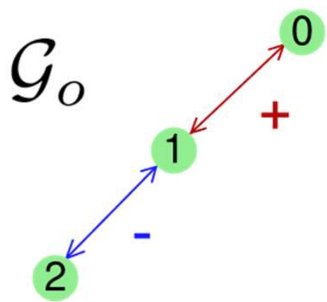


$$\mathbf{v}_a \approx (0.39, -0.80, 0.09, -0.45)$$

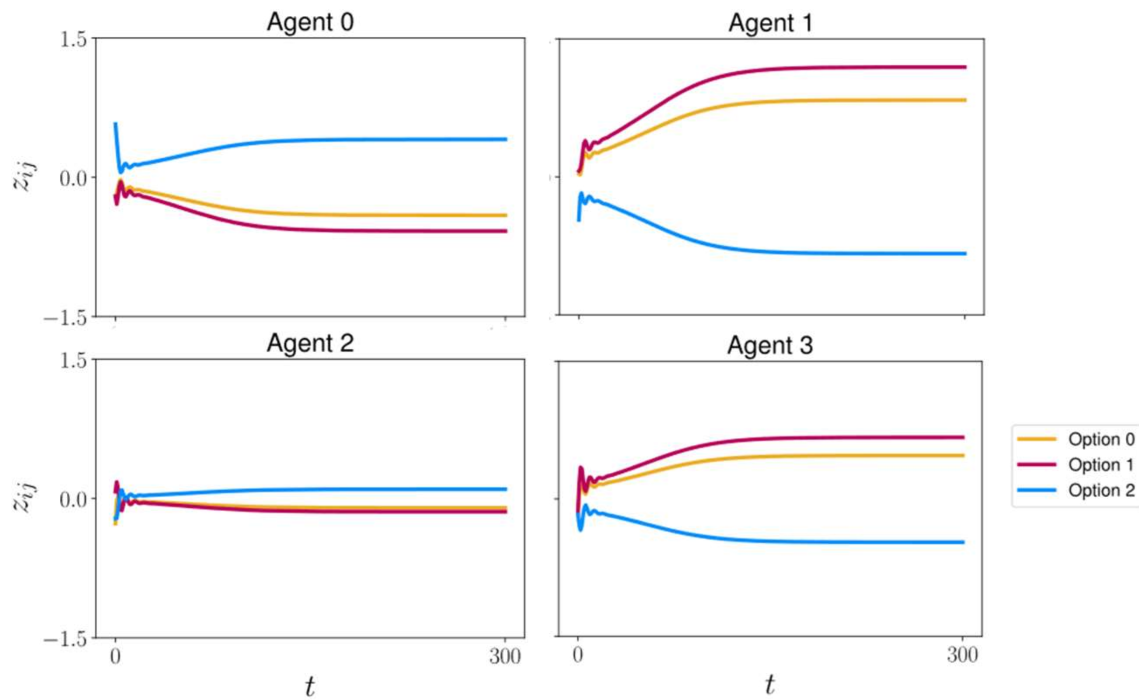


Role of graph structure: belief system

Bifurcation happens along span($\mathbf{v}_a \otimes \mathbf{v}_o$): eigenvector of belief system graph organizes internal beliefs of each agent



$$\mathbf{v}_o = \left(\frac{1}{2}, \frac{\sqrt{2}}{2}, -\frac{1}{2}\right)$$



Sensitivity to distributed biases: bifurcation unfolding

$$\dot{z}_{ij} = -d\, z_{ij} + u S_1\left(\alpha z_{ij} + \gamma \sum_{\substack{k=1\\k\neq i}}^{N_a} A^a_{ik} z_{kj}\right) + u \sum_{\substack{l=1\\l\neq j}}^{N_o} S_2\left(\beta A^o_{jl} z_{il} + \delta A^o_{jl} \sum_{\substack{k=1\\k\neq i}}^{N_a} A^a_{ik} z_{kl}\right) + b_{ij}$$

Proposition (Unfolding)

Suppose $\Lambda = \{(\lambda_a, \mu_o)\}$, $\alpha + \gamma\lambda_a + \beta\mu_o + \delta\lambda_a\mu_o > 0$, and S_1, S_2 have an odd symmetry. Let $\mathbf{v}_a, \mathbf{w}_a \in \mathbb{R}^{N_a}$ and $\mathbf{v}_o, \mathbf{w}_o \in \mathbb{R}^{N_o}$ be the right and left eigenvectors of A_a and A_o corresponding to λ_a and μ_o, respectively. Whenever

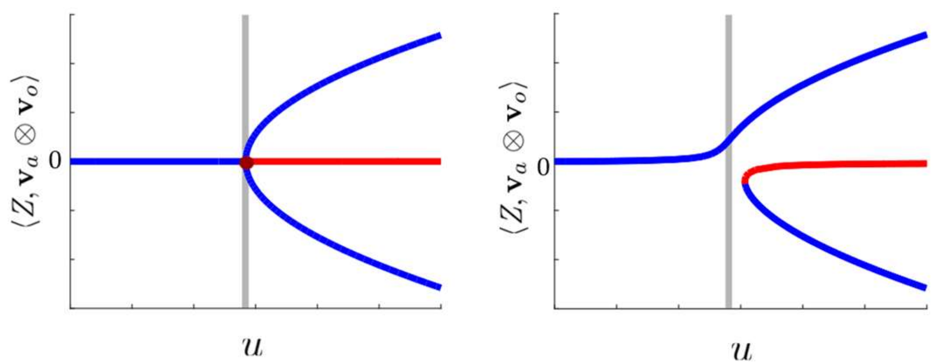
$$\langle \mathbf{w}_a \otimes \mathbf{w}_o, \mathbf{b} \rangle > 0 (< 0) \quad (1)$$

on a small neighborhood of u near u^* the local bifurcation diagram of the belief dynamics has a unique equilibrium which satisfies $\langle \mathbf{v}_a \otimes \mathbf{v}_o, \rangle > 0 (< 0)$.

Sensitivity to distributed biases: bifurcation unfolding

$$\dot{z}_{ij} = -d\, z_{ij} + u S_1\left(\alpha z_{ij} + \gamma \sum_{\substack{k=1 \\ k\neq i}}^{N_a} A^a_{ik} z_{kj}\right) + u \sum_{\substack{l=1 \\ l\neq j}}^{N_o} S_2\left(\beta A^o_{jl} z_{il} + \delta A^o_{jl} \sum_{\substack{k=1 \\ k\neq i}}^{N_a} A^a_{ik} z_{kl}\right) + b_{ij}$$

$\langle \mathbf{w}_a \otimes \mathbf{w}_o, \mathbf{b} \rangle = 0$ $\quad$ $\langle \mathbf{w}_a \otimes \mathbf{w}_o, \mathbf{b} \rangle > 0$

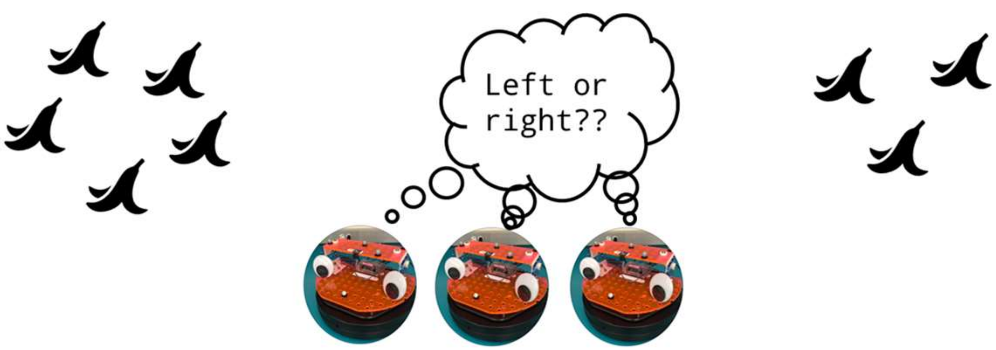


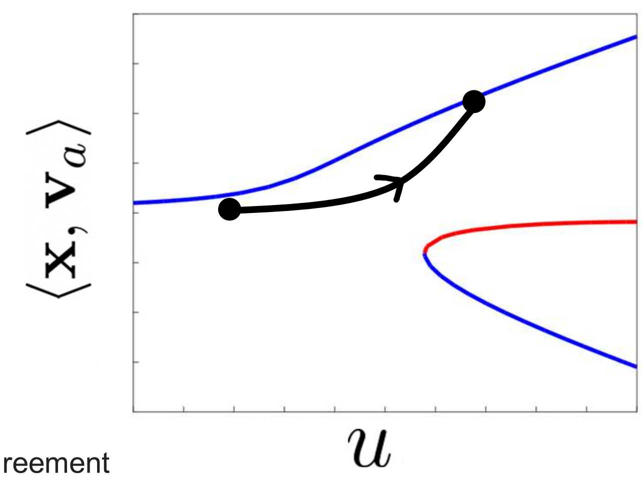
Sensitivity to small biases near bifurcation point

Design of social decisions in multi-agent systems

Dynamic attention allows group to be ultrasensitive to local information

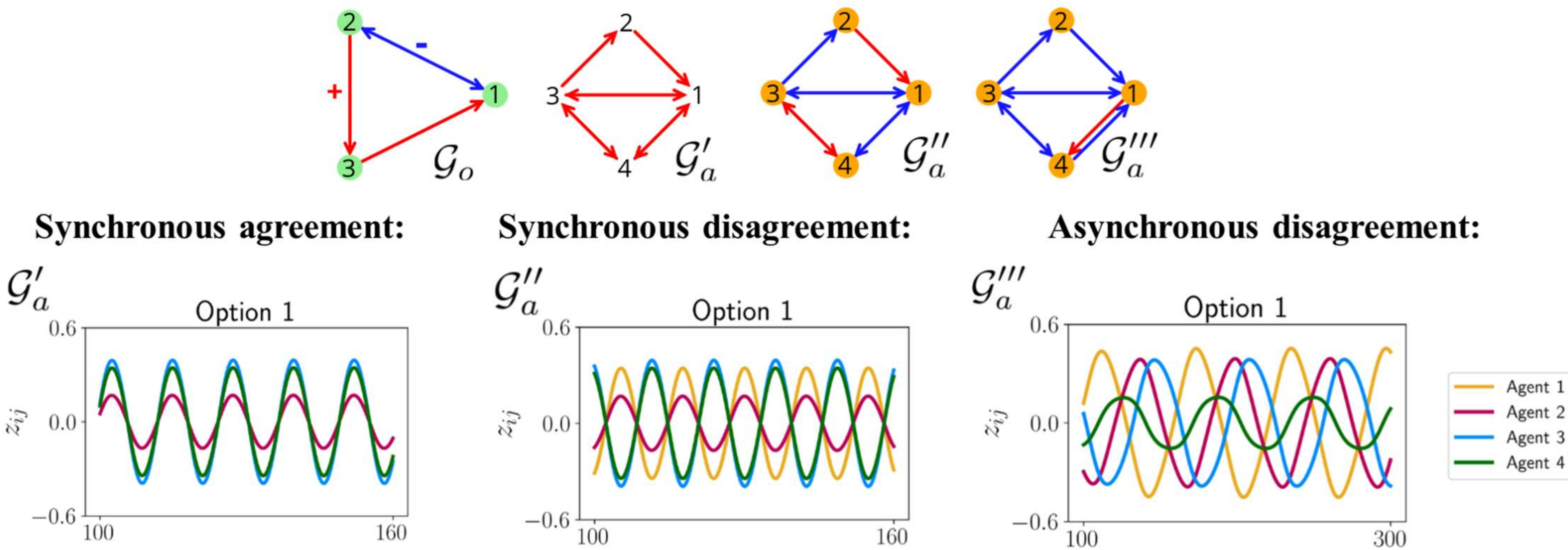
$$\dot{x}_i = -d_i x_i + u_i S\left(\alpha x_i + \gamma \sum_{\substack{k=1 \\ k \neq i}}^{N_a} A_{ik}^a x_k\right) + b_i \qquad \tau_u \dot{u}_i = -u_i + S_u\left(x_i^2 + \sum_{k=1}^{N_a} (a_{ik} x_k)^2\right)$$





A. Bizyaeva, T. Sorochkin, A. Franci, N.E. Leonard. "Control of agreement and disagreement cascades with distributed inputs." Proc. IEEE CDC, 2021.

Hopf bifurcation: belief oscillations



Period, relative phase, and relative amplitude of belief oscillations are related to leading eigenspaces of belief system and social network graphs

A. Bizyaeva, A. Franci, N.E. Leonard. Sustained oscillations in multi-topic belief dynamics over signed networks. *Proc. ACC, 2023*

Further results (check out our published and upcoming work!)

- Sufficient conditions for pitchfork and Hopf bifurcations based on structure of communication and belief system graphs; social imitation-driven and belief system-driven bifurcations

- Effect of external information or intrinsic biases (bifurcation unfolding)

- Bifurcations with symmetry in the graphs (symmetry breaking and synchrony breaking)

- Tunable flexibility and sensitivity in collective decisions with dynamic social parameters

- Applications: social network dynamics, flexible decision-making and task allocation for robotic teams, cognitive control allocation in individuals and groups

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References and acknowledgements

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