Conference Talk

The transition to synchronization of networked dynamical systems

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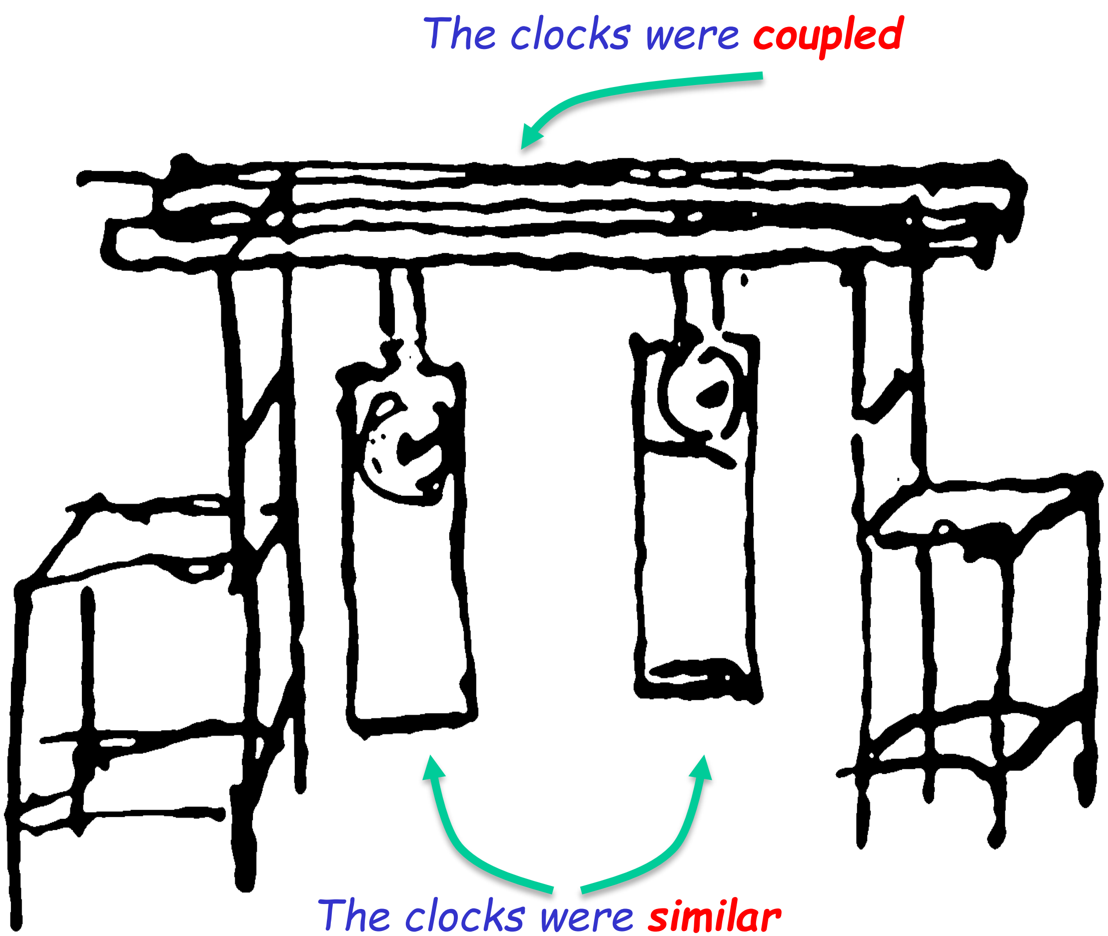
Synch is an old problem in physics: The *sympathetic* clocks of Huyghens

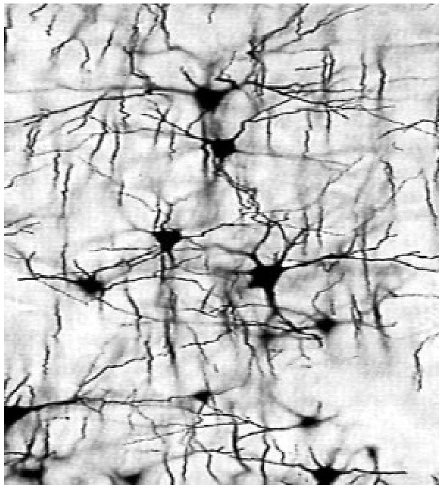


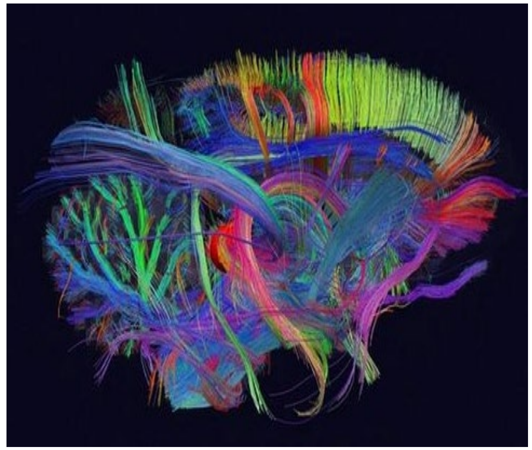
Christiaan Huyghens (1629-1695) discovered what he called **"an odd kind of sympathy"** between the clocks: regardless of their initial state, both *adopted the same rhythm*

Huygens correctly attributed the synchrony ***to tiny forces transmitted by the wooden beam*** from which they were suspended.

Synchronization in networked dynamical systems

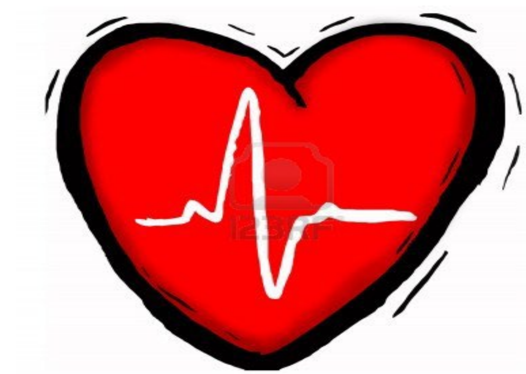
Synchronization of networked dynamical units is the collective behavior characterizing the functioning of most natural….





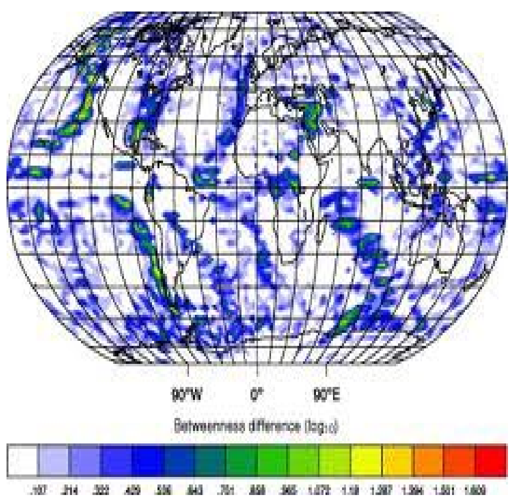
Brain dynamics

Heart beating



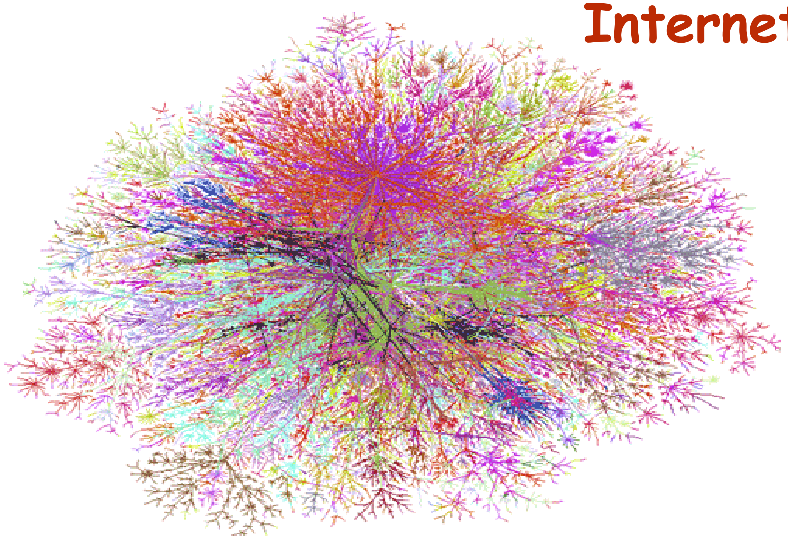
Animal behaviour

World clima ?



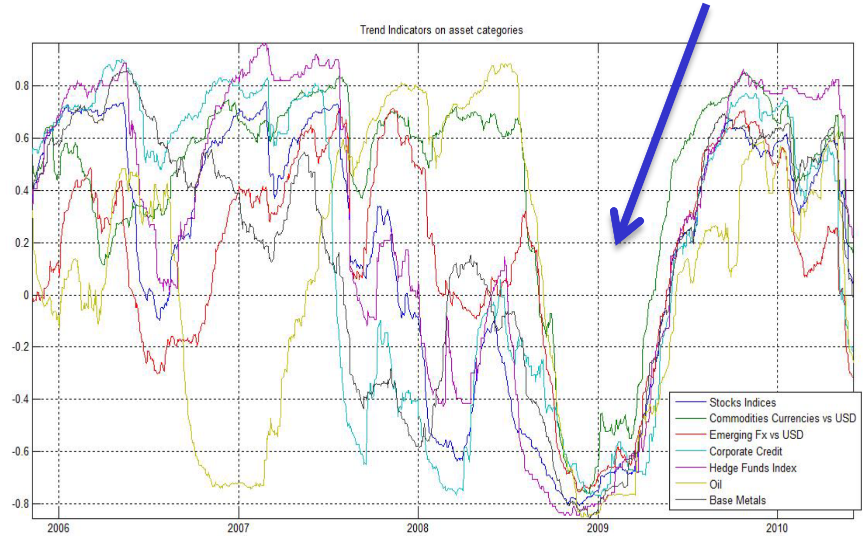
and man-made systems..

Internet



Financial markets

!?

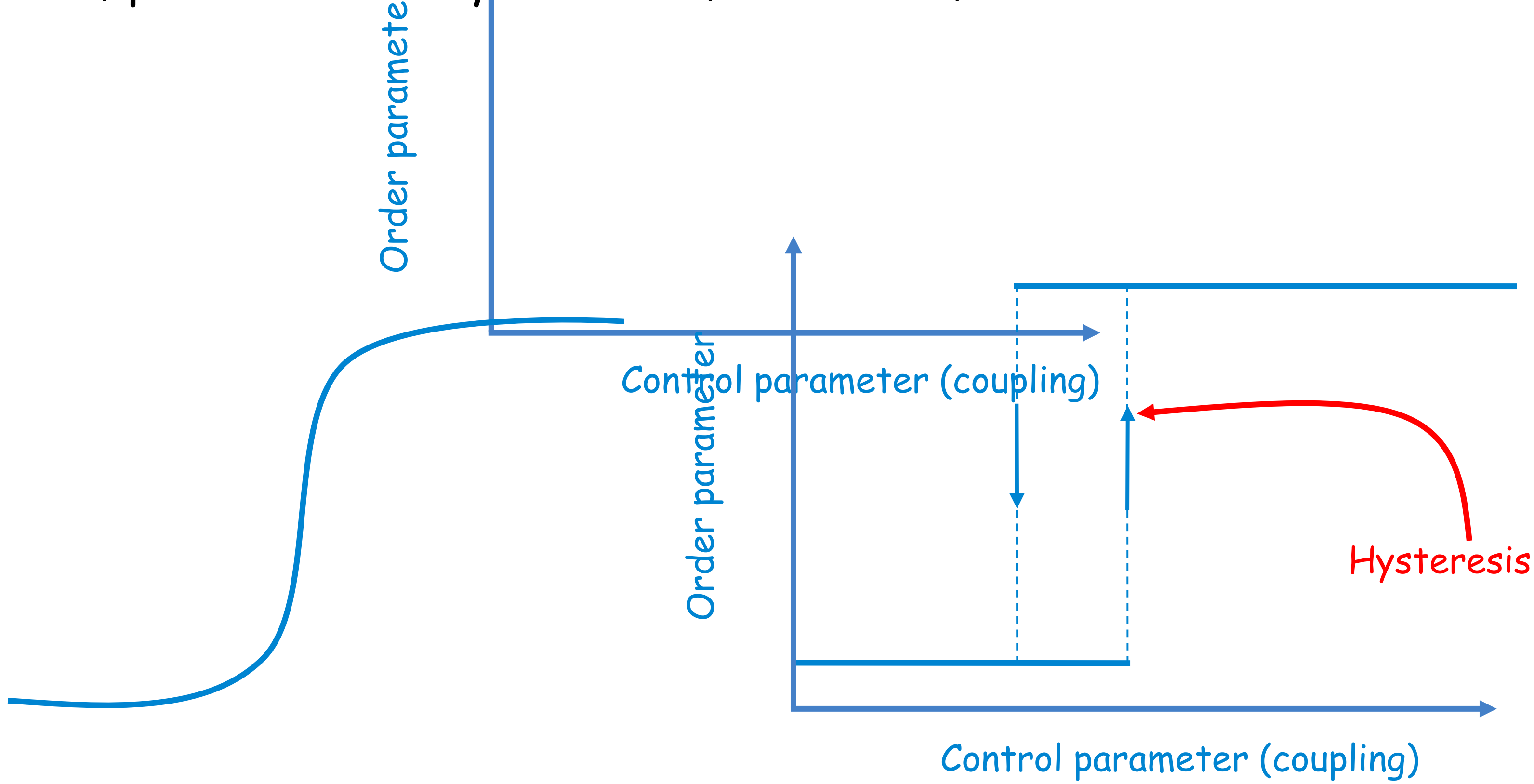


Human behaviour

Power grids

The transition to synchronization

Synchronization corresponds to a transition from a fully disordered, gaseous-like, phase to a fully ordered, solid-like, state



Second order-like transition (presence of intermediate phases)

First order-like transition (no intermediate phases)

GLOBAL SYNCHRONIZATION IN NETWORKS

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + \sigma \sum_{j=1}^{N} a_{ij}[\mathbf{h}(\mathbf{x}_j) - \mathbf{h}(\mathbf{x}_i)]$$

- N identical oscillators $\mathbf{x}_i \in \mathbb{R}^m$ with vector flow $\mathbf{f}$
- Oscillators coupled diffusively through the coupling function $\mathbf{h}$
- Adjacency matrix $a_{ij} = \begin{cases} 1 & \text{if i and j connected} \\ 0 & \text{otherwise} \end{cases}$
- σ global coupling parameter

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) - \sigma \sum_{j=1}^{N} \mathcal{L}_{ij}\mathbf{h}(\mathbf{x}_j)$$

- Laplacian matrix $\mathcal{L}_{ij}$ symmetric and zero row sum $\sum_{j=1}^{N} \mathcal{L}_{ij} = 0$

Master stability function

- Existence and invariance of the synchronization solution $\mathbf{x}_1(t) = \mathbf{x}_2(t) = \cdots = \mathbf{x}_N(t) = \mathbf{s}(t)$ obeying $\dot{\mathbf{s}} = \mathbf{f}(\mathbf{s})$ is warranted by the zero-row-sum property of $\mathcal{L}$.
- To study the stability of $\mathbf{s}$, one considers the perturbations $\delta\mathbf{x}_i(t) = \mathbf{x}_i(t) - \mathbf{s}(t)$ and write by the following linear (yet time dependent) equations

$$\delta\dot{\mathbf{x}}_i = J\mathbf{f}(\mathbf{s})\delta\mathbf{x}_i - \sigma\sum_{j=1}^{N}\mathcal{L}_{ij}J\mathbf{h}(\mathbf{s})\delta\mathbf{x}_j$$

  being $J\mathbf{f}$ and $J\mathbf{h}$ the corresponding Jacobian matrices of $\mathbf{f}$ and $\mathbf{h}$.
- In block form, one has $\delta\dot{\mathbf{x}} = \left[\mathbb{I}_N \otimes J\mathbf{f}(\mathbf{s}) - \sigma\mathcal{L} \otimes J\mathbf{h}(\mathbf{s})\right]\delta\mathbf{x}$, where $\delta\mathbf{x}$ is the following $m \cdot N \times 1$ vector

$$\delta\mathbf{x} = \left(\delta x_{11} \ldots \delta x_{m1}, \delta x_{12} \ldots \delta x_{m2}, \ldots \ldots \delta x_{1N} \ldots \delta x_{mN}\right)^t \quad .$$

MASTER STABILITY FUNCTION

- As $\mathcal{L}$ is zero-row sum and symmetric, it is diagonalizable, and if one orders by size its N eigenvalues λ_i ($0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$), one has that $\lambda_1 = 0$ with associated eigenvector $\mathbf{v}_1 \equiv \frac{1}{\sqrt{N}}(1, 1, 1, ..., 1)^T$ which defines the synchronization manifold!
- All other eigenvectors $\mathbf{v}_i$ of $\mathcal{L}$ form a basis of the space tangent to the synchronization manifold!
- The perturbation vector $\delta\mathbf{x}$ can be expanded on the orthonormal basis formed by $\{\mathbf{v}_i\}$ as

$$\delta\mathbf{x} = \sum_{i=1}^{N} \mathbf{v}_i \otimes \xi_i.$$

- Substituting the expansion in the linearized equation, and applying $\mathbf{v}_i^t \otimes \mathbb{I}_m$ to the left side of each term, one obtains equations:

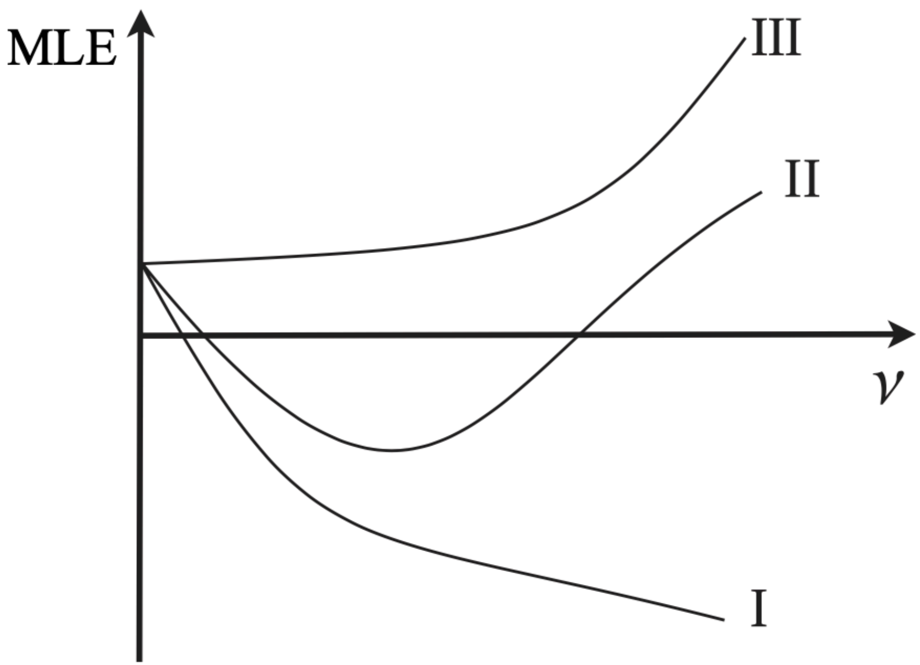
$$\dot{\xi}_i = [J\mathbf{f}(\mathbf{s}) - \sigma\lambda_i J\mathbf{h}(\mathbf{s})]\xi_i.$$

Master stability function

- The previous equation can be rewritten as a parametric equation ($\nu = \sigma\lambda$):

$$\dot{\xi} = [J\mathbf{f}(\mathbf{s}) - \nu J\mathbf{h}(\mathbf{s})]\xi$$

- The maximum Lyapunov exponent ($MLE(\nu)$) defines the Master Stability Function and its negativity implies that the synchronization manifold is stable.

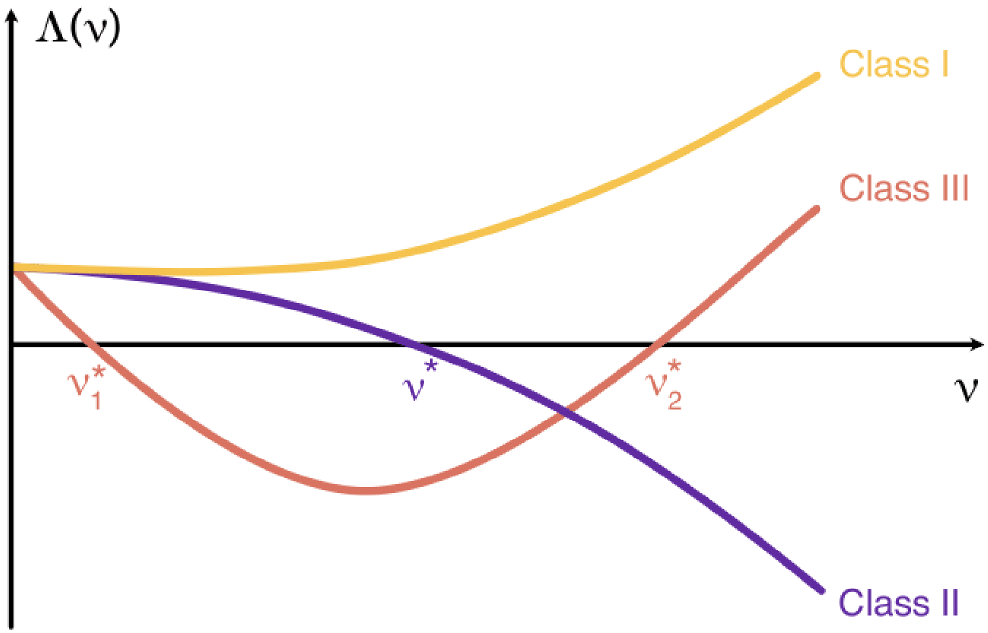


- Class I: $MLE < 0$ for $\nu > \nu_m$, and $\mathbf{s}$ is stable for $\sigma > \nu_m/\lambda_2$
- Class II: $MLE < 0$ for $\nu \in [\nu_m, \nu_M]$ and $\mathbf{s}$ is stable if $\lambda_N/\lambda_2 < \sigma < \nu_M/\nu_m$
- Class III: $MLE > 0\ \forall\nu$, the synchronous solution is never stable

Complex networks: Structure and dynamics, S. Boccaletti et al., Phys Rep 424, 175-308 (2006)

Unveiling the path to synchrony (I)

Class I systems defy synchronization.
Neither the synchronized solution nor any other cluster-synchronized state will ever be stable



In Class II systems $d\,\lambda_2 > \nu^*$ warrants stability of synchronization.

In Class III systems, the entire spectrum of Laplacian eigenvalues must fall (when multiplied by d) in between ν^*_1 and ν^*_2.
The two conditions $d\,\lambda_N < \nu^*_2$ and $d\,\lambda_2 > \nu^*_1$ must be verified.
The former condition gives a bound for the coupling strength the latter provides once again the threshold for synchronization.

Has "synchronizability" any sense?

Unfortunately, the attention concentrated on a quantity that was called ***synchronizability***, given by the ratio

$$R = \frac{\lambda_N}{\lambda_2}$$

between the max and the second smallest eigenvalue of the Laplacian.

Does it make any sense?

For Class I systems it is just senseless.

For Class II systems it is even wrong… (the range of coupling strength for which synchronization is stable is unbounded), and the threshold only depends on Λ_2 .
Counterexample: take two graphs G_1 and G_2 such that $\Lambda_2(G_1) = 1$, $\Lambda_N(G_1) = 2$ and $\Lambda_2(G_2) = 10^{45}$, $\Lambda_N(G_2) = 10^{46}$. According to R, G_1 is more synchronizable than G_2 , but the threshold for synch of G_2 is 45 orders of magnitude (!!!) smaller than that of G_1 .

Only for Class III systems, there is some sense to R, but ONLY to indicate the range of coupling strength for which synch persists in the stable region, since the threshold is still depending only on Λ_2.

Unveiling the path to synchrony (II)

There are three conceptual steps that need to be made.

First step (unfolding the trasverse space)

- As d progressively increases, the eigenvalues λ_i cross the critical point sequentially.
  All eigenvalues will cross the critical point one by one (if not degenerate) in the reverse order of their size.

- At each value of d one can consider the subspace T (d) having as orthonomal basis the set of eigenvectors $\{\mathbf{v}_i\}$ whose corresponding λ_i (multiplied by d) have already crossed the stability threshold.
  Therefore, T (d) will **ALWAYS** (i.e., at all values of d) contain only contracting directions.

Unveiling the path to synchrony (III)

The second step (examining eigenvector componentwise!)

If one constructs the matrix V having as columns the eigenvectors

$$V_{N\times N} = \begin{bmatrix} v_{1,1} & v_{2,1} & \cdots & v_{N,1} \\ v_{1,2} & v_{2,2} & \cdots & v_{N,2} \\ \vdots & \vdots & \cdots & \vdots \\ v_{1,N} & v_{2,N} & \cdots & v_{N,N} \end{bmatrix}$$

then the rows of V provide an orthonormal basis as well!!

This is because the columns of V are an orthonormal basis, implying that $V V^T = I$ or, equivalently, that $V^T = V^{-1}$. Therefore, $I = V^{-1} V = V^T V$.

The relevant consequence is that one can now examine the eigenvectors componentwise!!!

Unveiling the path to synchrony (IV)

The E_{λ_i} and S_N matrices

In particular, for each λ_i, one can consider

$$V_i = [v_{i,1} \quad v_{i,2} \quad \cdots \quad v_{i,N}]^T$$

$$E_{\lambda_i} = \begin{bmatrix} (v_{i,1} - v_{i,1})^2 & (v_{i,2} - v_{i,1})^2 & \cdots & (v_{i,N} - v_{i,1})^2 \\ (v_{i,1} - v_{i,2})^2 & (v_{i,2} - v_{i,2})^2 & \cdots & (v_{i,N} - v_{i,2})^2 \\ \vdots & \vdots & \cdots & \vdots \\ (v_{i,1} - v_{i,N})^2 & (v_{i,2} - v_{i,N})^2 & \cdots & (v_{i,N} - v_{i,N})^2 \end{bmatrix}_{N\times N}$$

These matrices are symmetric, and the diagonal elements are equal to zero.

Then, initialize S_{N+1} with a zero matrix, and, for $i = N{:}-1{:}1$, do

$$S_i = S_{i+1} + E_{\lambda_i}$$

At the end of this step, one has $S_N, S_{N-1}, \ldots S_1$

The properties of the S_N matrices.

- As $\mathbf{v}_1$ is aligned with the synchronization manifold M, all its components are equal, and therefore E_{λ_1} =0 and S_1=S_2.

- All diagonal elements of all S matrices are zero.

- The off diagonal (ij) elements of the matrix S_n (n=1,…,N) are nothing but **the square of the norm of the vector obtained as the difference between the two 1-norm vectors defined by rows i and j of matrix V**, limited to their n last components.

- As so, the maximum value that any entry (ij) may have in matrices S_n is 2, which corresponds to the case in which such two vectors are orthogonal.

- For what said above, all off-diagonal entries of S_2 are equal to 2.

Third step (localized spectral blocks)

The third step consists in considering the fact that the Laplacian matrix L uniquely defines G, and as so any clustering property of G should be reflected into a corresponding spectral feature of L.

Definition

A subset $S_{(i1,\dots,ik)}$ consisting of k-1 eigenvectors forms a spectral block localized at nodes $(i_1, \dots, i_k)$ if

- each eigenvector belonging to the subset has all entries (except $i_1, \dots, i_k$) equal to 0;
- for each other eigenvector not belonging to the subset, the entries $i_1, \dots, i_k$ are all equal

Moreover, all eigenvectors ($v_2, v_3, \dots, v_N$) are orthogonal to v_1, and therefore the sum of all their entries must be equal to 0.

This allows to demonstrate the Theorem stated below:

Theorem. The 2 following statements are equivalent:

1. All k nodes belonging to a cluster defined by the indices $(i_1, \ldots, i_k)$ have the same connections with the same weights with all other nodes not belonging to the cluster i.e., for any $(p, q) \in (i_1, \ldots, i_k)$ and $j \notin (i_1, \ldots, i_k)$ one has $L_{pj} = L_{qj}$.

2. There is a spectral block $S_{(i1,\ldots,ik)}$ made of k-1 Laplacian's eigenvectors localized at nodes $(i_1, \ldots, i_k)$

Unveiling the path to synchrony (VIII)

Consequences of the theorem

- The matrices S_n may have entries equal to 2 also for n >2 (when a subset of eigenvectors unfolding T forms a localized spectral block).

- Conceptually, the nodes belonging to a given cluster are indistinguishable to the eyes of any other node of the network, they receive an equal input from the rest of the network, and therefore (for the principle that a same input will eventually - i.e., at sufficiently large coupling -imply a same output) they may synchronize independently on the synchronization properties of the rest of the graph.

Unveiling the path to synchrony (IX)

- The theorem puts no constraints on the way nodes are connected within the cluster. Therefore, fulfillment of the theorem is realized by (but is not limited to) the network's symmetry orbits.

- The situation is therefore that:
  - a) all symmetry orbits in graph G give rise to clusters that may synchronize during the transition;
  - b) the condition for clusters to synchronize is more general than constituting a symmetry orbit: the only requirement is that they receive an equal input from the rest of the network;
  - c) clusters that are being formed in the transition constitute specific (external) equitable partitions of G

- Therefore, our study clarifies once forever that the intermediate structured states in the path to synchrony of a network are **more general** than the graph's symmetry orbit, but **more specific** than the graph's equitable partitions.

Unveiling the path to synchrony (X)

Finally, we can …cook the cake!

The algorithm to completely describe the path to synchronization consists in the following steps:

- given a network G, one considers the Laplacian matrix L, and extracts its N eigenvalues λ_i (ordered in size) and the corresponding eigenvectors $\mathbf{v}_i$. One then calculates the matrices $E_{\lambda i}$ and S_n;
- one inspects the matrices S_n in the same order with which the Laplacian's eigenvalues (when multiplied by d) crosses the critical point (N, N-1, N-2, ..., 2, 1), and looks for entries which are equal to 2;
- when, for the first time in the sequence (say, for index p) an entry in matrix S_p is (or multiple entries are) found equal to 2, a prediction is made that an event will occur in the transition: the cluster (or clusters) formed by the nodes with labels equal to those of the found entry (entries) will synchronize at the coupling strength value ν^*/λ_p . The inspection of matrices S_n then continues, focusing only on the entries different from those already found to be 2 at level S_p;
- once having inspected all S_n matrices, one obtains therefore the complete description of the sequence of events occurring in the transition, with the exact indication of all the values of the critical coupling strengths at which each of such events is occurring.

An Illustration: Fully connected weighted network

N=10

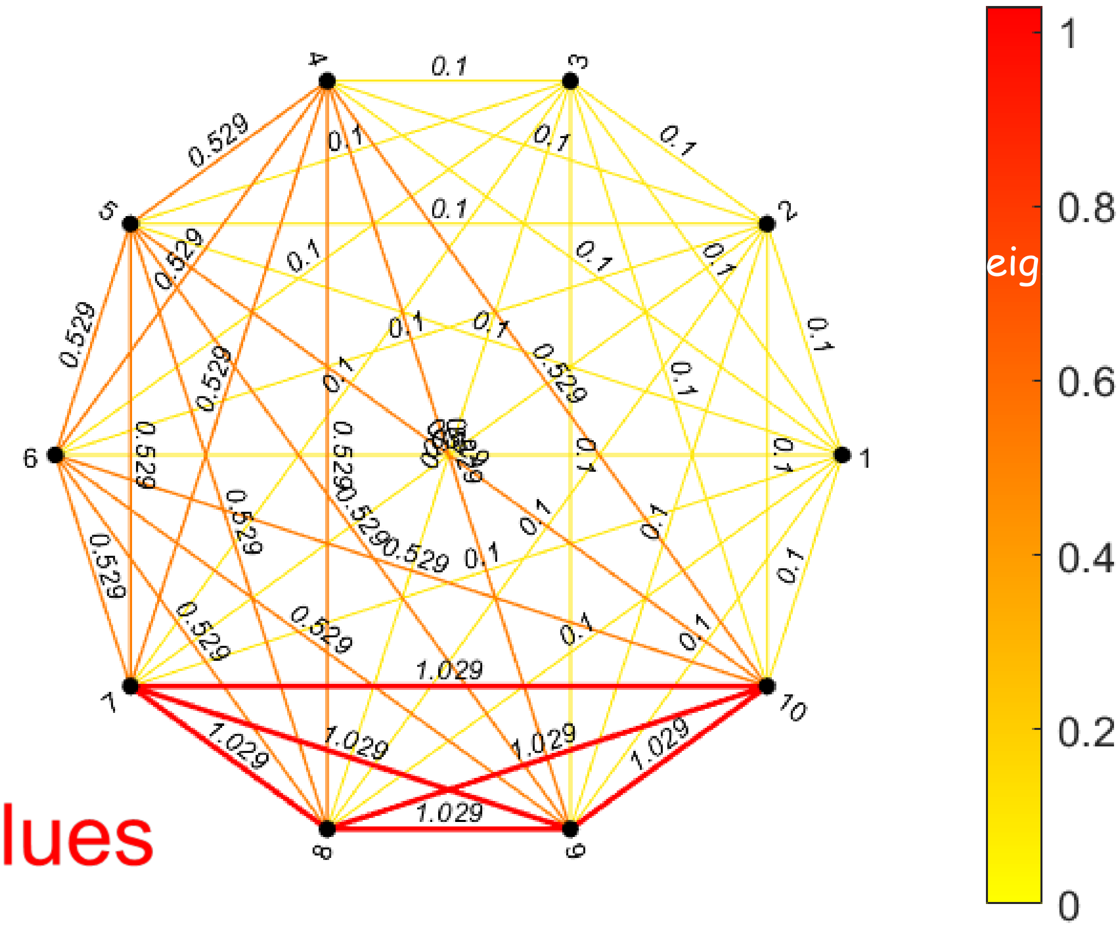
Clusters:

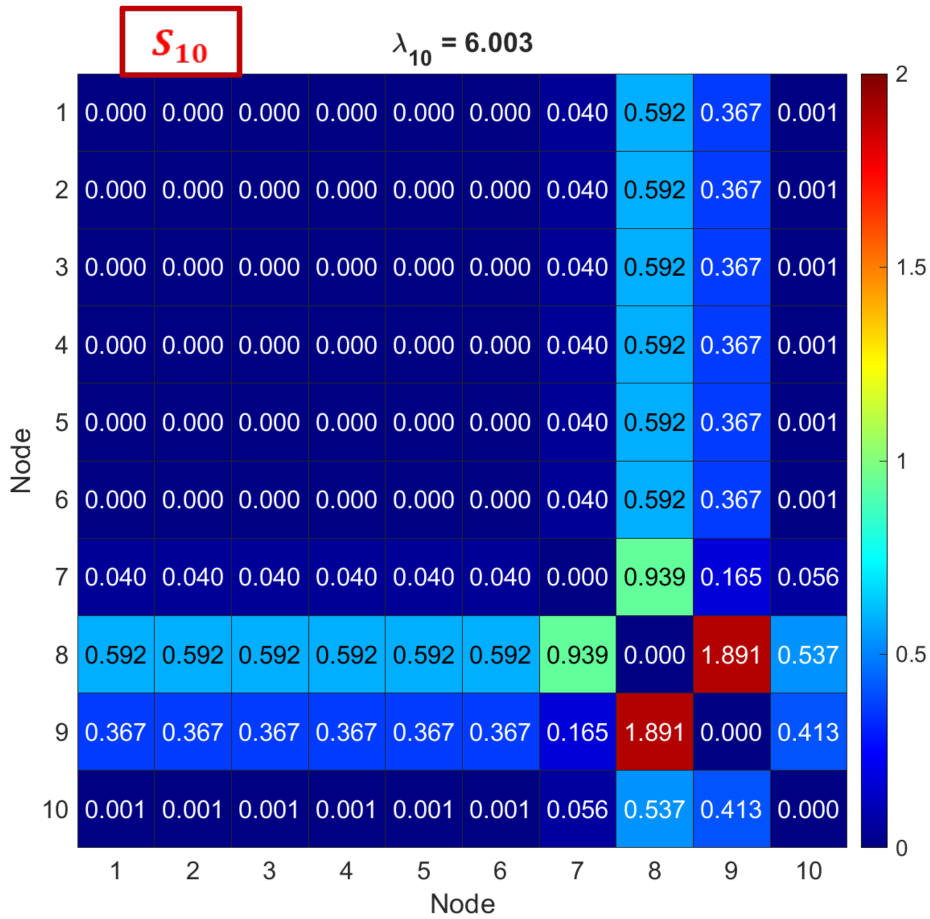
{10,9,8,7}

{6,5,4}

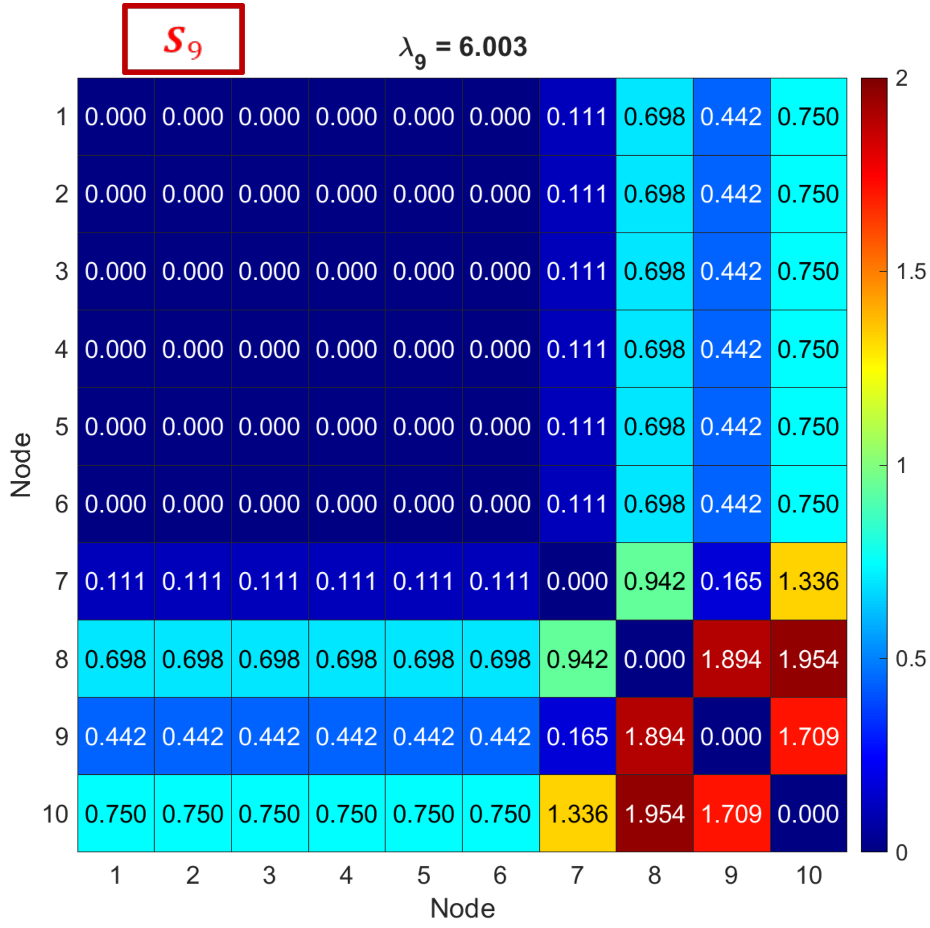
{3,2,1}

Laplacian eigenvalues

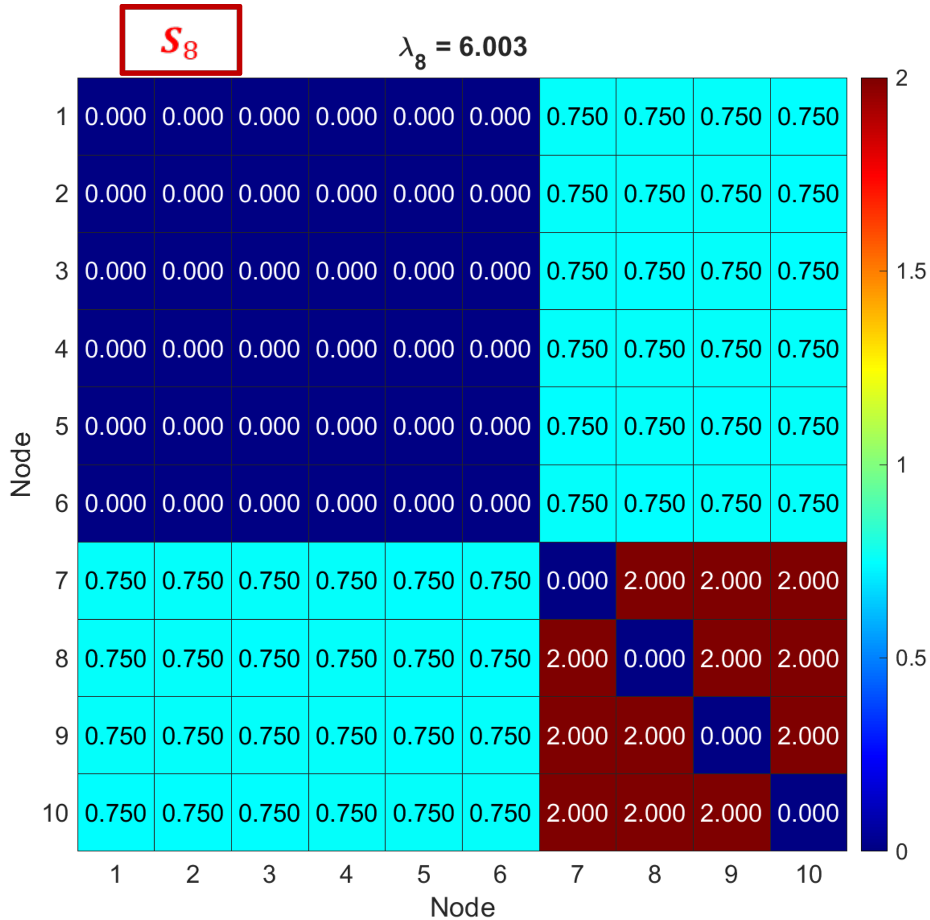




S_{10}
λ_{10} = 6.003
Node
Node
1 0.000 0.000 0.000 0.000 0.000 0.000 0.040 0.592 0.367 0.001
2 0.000 0.000 0.000 0.000 0.000 0.000 0.040 0.592 0.367 0.001
3 0.000 0.000 0.000 0.000 0.000 0.000 0.040 0.592 0.367 0.001
4 0.000 0.000 0.000 0.000 0.000 0.000 0.040 0.592 0.367 0.001
5 0.000 0.000 0.000 0.000 0.000 0.000 0.040 0.592 0.367 0.001
6 0.000 0.000 0.000 0.000 0.000 0.000 0.040 0.592 0.367 0.001
7 0.040 0.040 0.040 0.040 0.040 0.040 0.000 0.939 0.165 0.056
8 0.592 0.592 0.592 0.592 0.592 0.592 0.939 0.000 1.891 0.537
9 0.367 0.367 0.367 0.367 0.367 0.367 0.165 1.891 0.000 0.413
10 0.001 0.001 0.001 0.001 0.001 0.001 0.056 0.537 0.413 0.000
1 2 3 4 5 6 7 8 9 10
2
1.5
1
0.5
0



S_9
λ_9 = 6.003
Node
Node
1 0.000 0.000 0.000 0.000 0.000 0.000 0.111 0.698 0.442 0.750
2 0.000 0.000 0.000 0.000 0.000 0.000 0.111 0.698 0.442 0.750
3 0.000 0.000 0.000 0.000 0.000 0.000 0.111 0.698 0.442 0.750
4 0.000 0.000 0.000 0.000 0.000 0.000 0.111 0.698 0.442 0.750
5 0.000 0.000 0.000 0.000 0.000 0.000 0.111 0.698 0.442 0.750
6 0.000 0.000 0.000 0.000 0.000 0.000 0.111 0.698 0.442 0.750
7 0.111 0.111 0.111 0.111 0.111 0.111 0.000 0.942 0.165 1.336
8 0.698 0.698 0.698 0.698 0.698 0.698 0.942 0.000 1.894 1.954
9 0.442 0.442 0.442 0.442 0.442 0.442 0.165 1.894 0.000 1.709
10 0.750 0.750 0.750 0.750 0.750 0.750 1.336 1.954 1.709 0.000
1 2 3 4 5 6 7 8 9 10
2
1.5
1
0.5
0



S_8
λ_8 = 6.003
Node
Node
1 2 3 4 5 6 7 8 9 10
0.000 0.000 0.000 0.000 0.000 0.000 0.750 0.750 0.750 0.750
0.000 0.000 0.000 0.000 0.000 0.000 0.750 0.750 0.750 0.750
0.000 0.000 0.000 0.000 0.000 0.000 0.750 0.750 0.750 0.750
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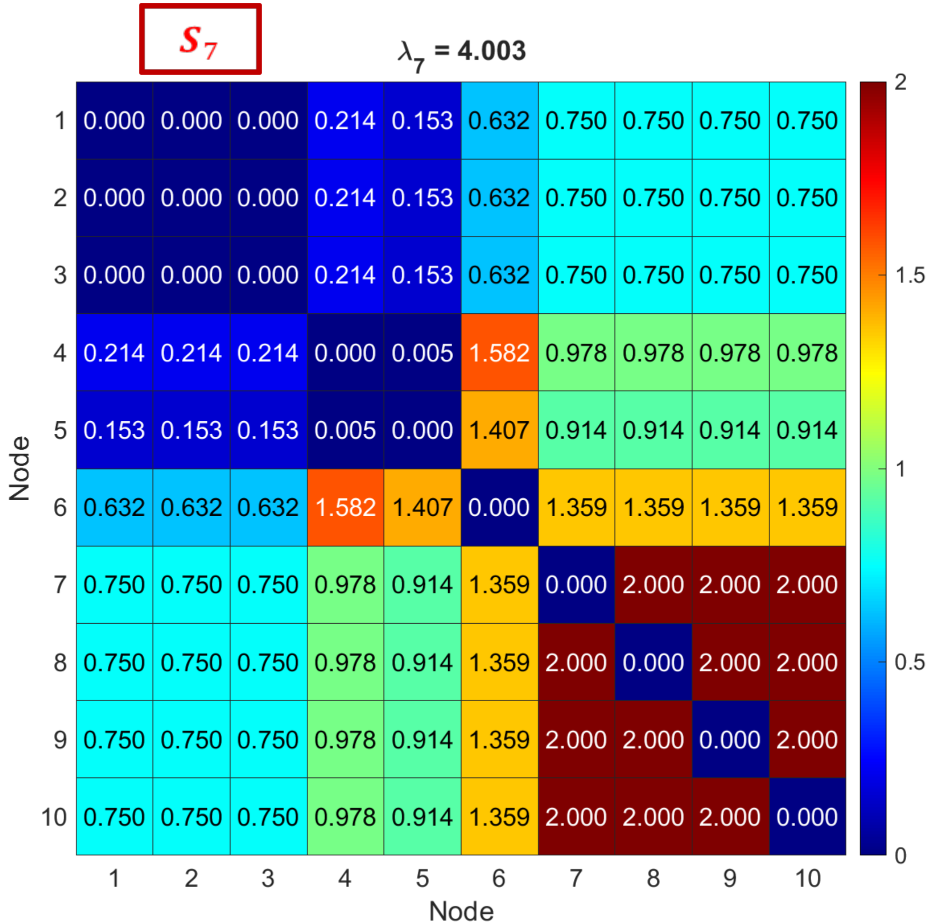
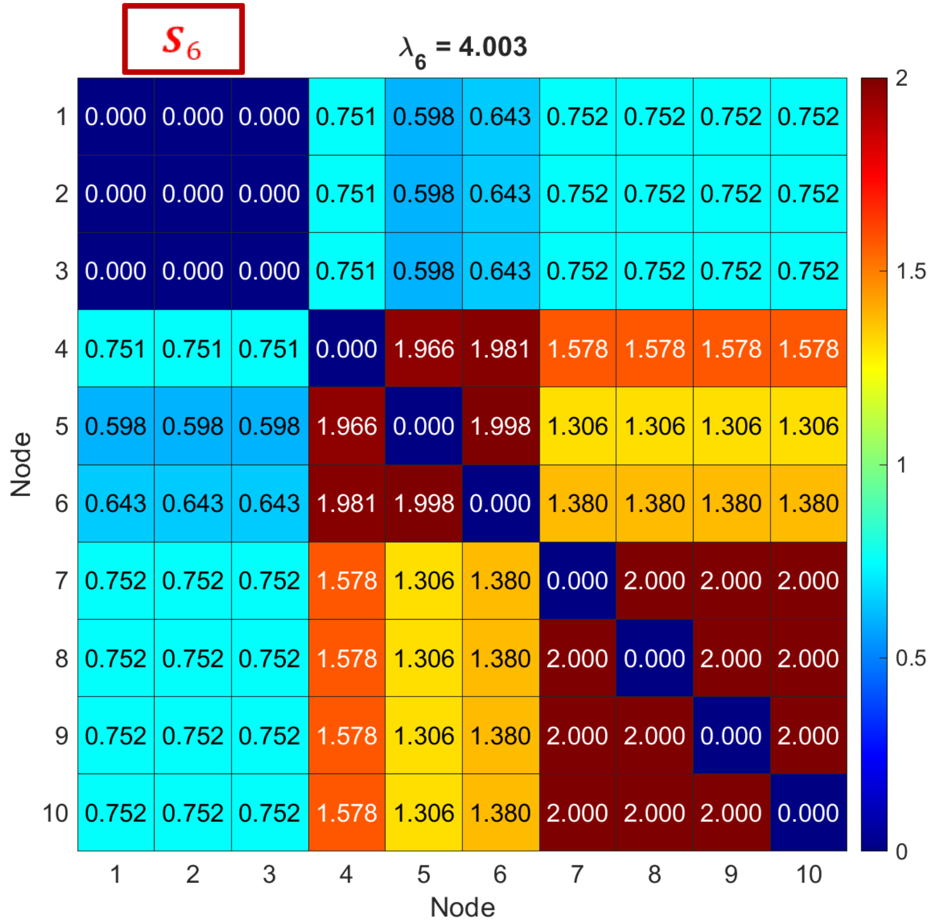
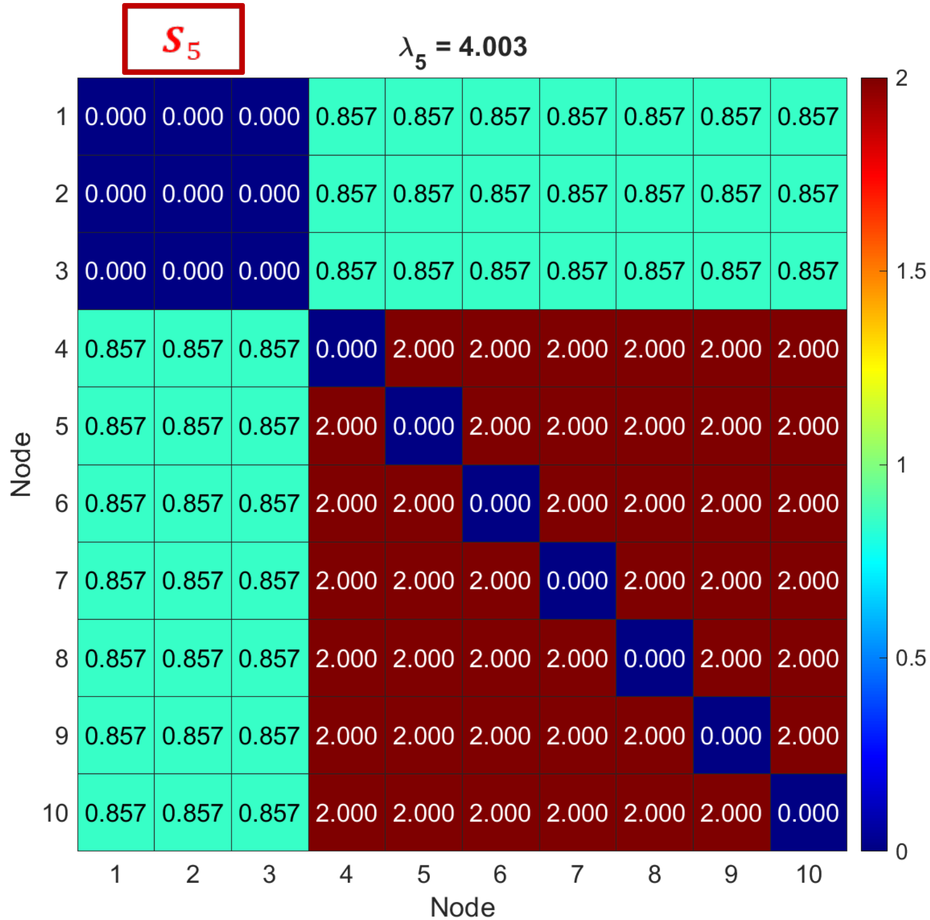
0 0.5 1 1.5 2



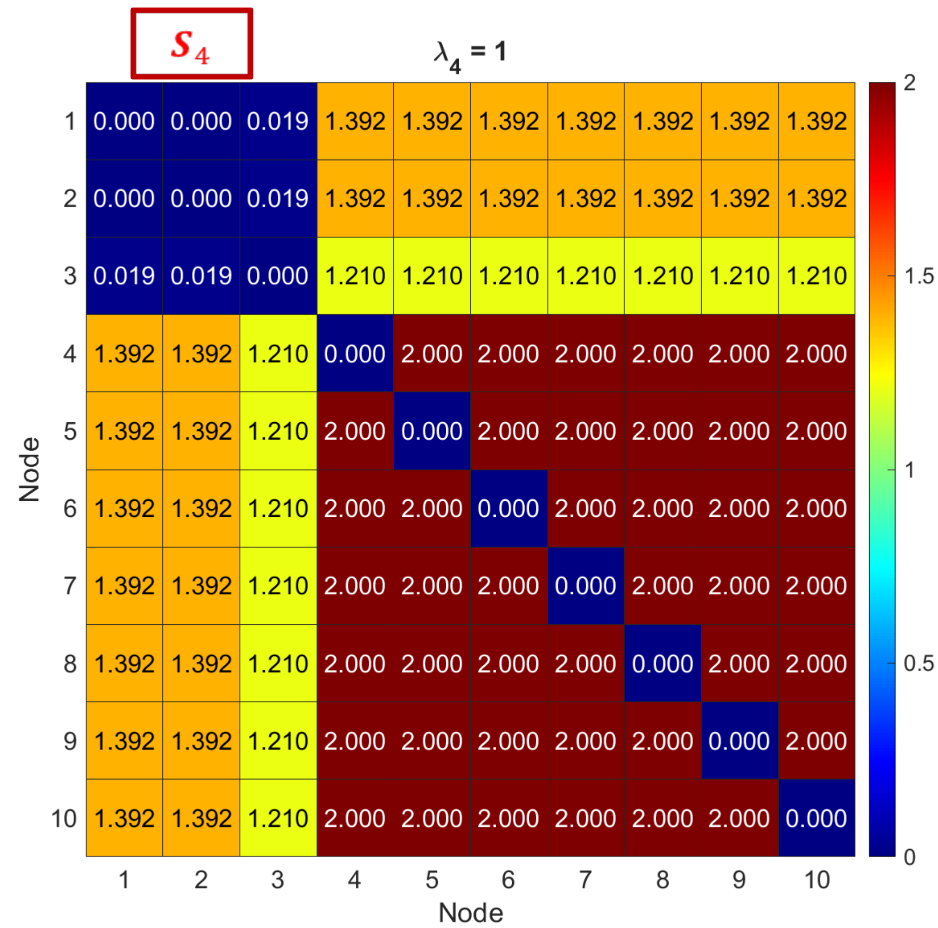
S_7
λ_7 = 4.003
Node
Node
1 0.000 0.000 0.000 0.214 0.153 0.632 0.750 0.750 0.750 0.750
2 0.000 0.000 0.000 0.214 0.153 0.632 0.750 0.750 0.750 0.750
3 0.000 0.000 0.000 0.214 0.153 0.632 0.750 0.750 0.750 0.750
4 0.214 0.214 0.214 0.000 0.005 1.582 0.978 0.978 0.978 0.978
5 0.153 0.153 0.153 0.005 0.000 1.407 0.914 0.914 0.914 0.914
6 0.632 0.632 0.632 1.582 1.407 0.000 1.359 1.359 1.359 1.359
7 0.750 0.750 0.750 0.978 0.914 1.359 0.000 2.000 2.000 2.000
8 0.750 0.750 0.750 0.978 0.914 1.359 2.000 0.000 2.000 2.000
9 0.750 0.750 0.750 0.978 0.914 1.359 2.000 2.000 0.000 2.000
10 0.750 0.750 0.750 0.978 0.914 1.359 2.000 2.000 2.000 0.000
1 2 3 4 5 6 7 8 9 10
0 0.5 1 1.5 2



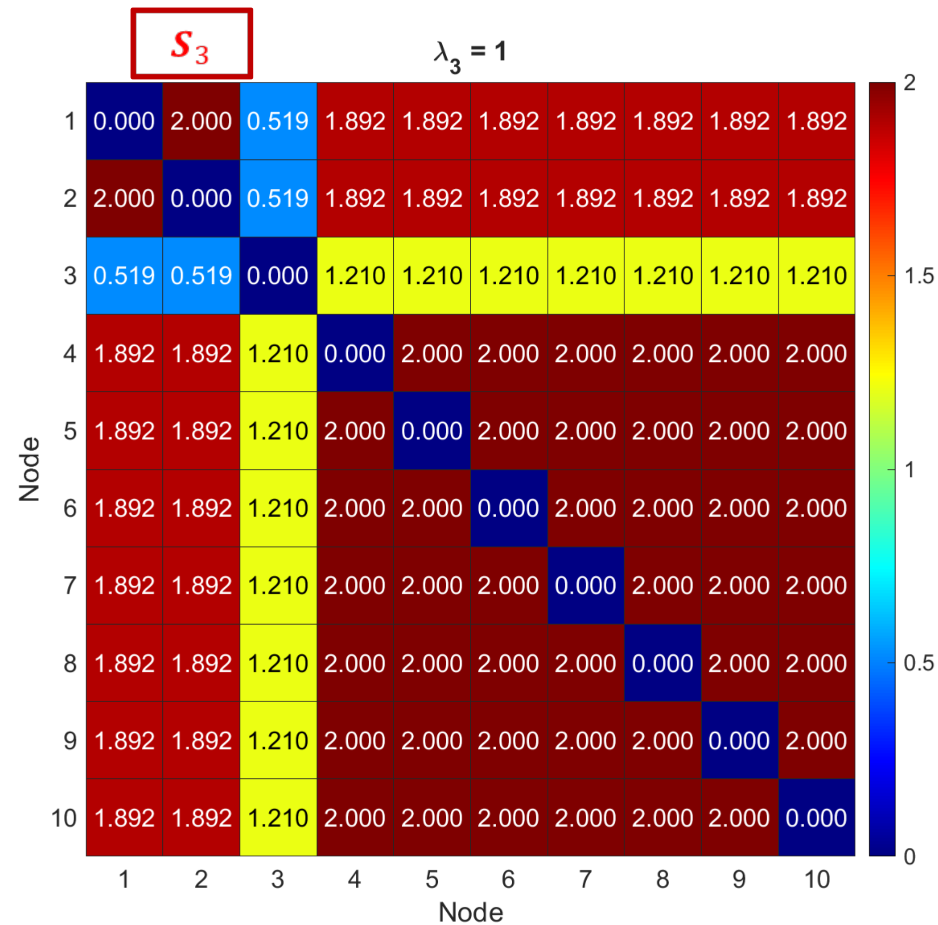
s_6
λ_6 = 4.003
1 0.000 0.000 0.000 0.751 0.598 0.643 0.752 0.752 0.752 0.752
2 0.000 0.000 0.000 0.751 0.598 0.643 0.752 0.752 0.752 0.752
3 0.000 0.000 0.000 0.751 0.598 0.643 0.752 0.752 0.752 0.752
4 0.751 0.751 0.751 0.000 1.966 1.981 1.578 1.578 1.578 1.578
5 0.598 0.598 0.598 1.966 0.000 1.998 1.306 1.306 1.306 1.306
6 0.643 0.643 0.643 1.981 1.998 0.000 1.380 1.380 1.380 1.380
7 0.752 0.752 0.752 1.578 1.306 1.380 0.000 2.000 2.000 2.000
8 0.752 0.752 0.752 1.578 1.306 1.380 2.000 0.000 2.000 2.000
9 0.752 0.752 0.752 1.578 1.306 1.380 2.000 2.000 0.000 2.000
10 0.752 0.752 0.752 1.578 1.306 1.380 2.000 2.000 2.000 0.000
1 2 3 4 5 6 7 8 9 10
Node
Node
2
1.5
1
0.5
0



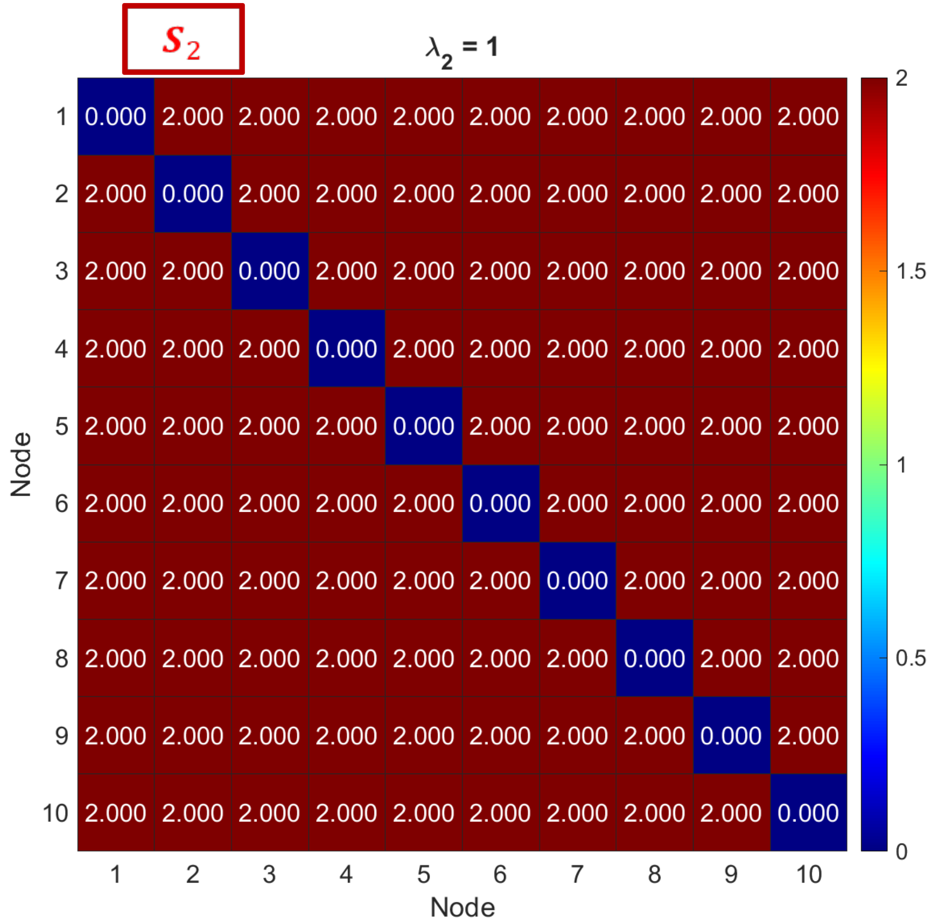
S_5
λ_5 = 4.003
1 0.000 0.000 0.000 0.857 0.857 0.857 0.857 0.857 0.857 0.857
2 0.000 0.000 0.000 0.857 0.857 0.857 0.857 0.857 0.857 0.857
3 0.000 0.000 0.000 0.857 0.857 0.857 0.857 0.857 0.857 0.857
4 0.857 0.857 0.857 0.000 2.000 2.000 2.000 2.000 2.000 2.000
5 0.857 0.857 0.857 2.000 0.000 2.000 2.000 2.000 2.000 2.000
6 0.857 0.857 0.857 2.000 2.000 0.000 2.000 2.000 2.000 2.000
7 0.857 0.857 0.857 2.000 2.000 2.000 0.000 2.000 2.000 2.000
8 0.857 0.857 0.857 2.000 2.000 2.000 2.000 0.000 2.000 2.000
9 0.857 0.857 0.857 2.000 2.000 2.000 2.000 2.000 0.000 2.000
10 0.857 0.857 0.857 2.000 2.000 2.000 2.000 2.000 2.000 0.000
1 2 3 4 5 6 7 8 9 10
Node
Node
2
1.5
1
0.5
0



S_4
λ_4 = 1
1 0.000 0.000 0.019 1.392 1.392 1.392 1.392 1.392 1.392 1.392
2 0.000 0.000 0.019 1.392 1.392 1.392 1.392 1.392 1.392 1.392
3 0.019 0.019 0.000 1.210 1.210 1.210 1.210 1.210 1.210 1.210
4 1.392 1.392 1.210 0.000 2.000 2.000 2.000 2.000 2.000 2.000
5 1.392 1.392 1.210 2.000 0.000 2.000 2.000 2.000 2.000 2.000
6 1.392 1.392 1.210 2.000 2.000 0.000 2.000 2.000 2.000 2.000
7 1.392 1.392 1.210 2.000 2.000 2.000 0.000 2.000 2.000 2.000
8 1.392 1.392 1.210 2.000 2.000 2.000 2.000 0.000 2.000 2.000
9 1.392 1.392 1.210 2.000 2.000 2.000 2.000 2.000 0.000 2.000
10 1.392 1.392 1.210 2.000 2.000 2.000 2.000 2.000 2.000 0.000
1 2 3 4 5 6 7 8 9 10
Node
Node
2
1.5
1
0.5
0



S3
λ3 = 1
Node
Node
1 0.000 2.000 0.519 1.892 1.892 1.892 1.892 1.892 1.892 1.892
2 2.000 0.000 0.519 1.892 1.892 1.892 1.892 1.892 1.892 1.892
3 0.519 0.519 0.000 1.210 1.210 1.210 1.210 1.210 1.210 1.210
4 1.892 1.892 1.210 0.000 2.000 2.000 2.000 2.000 2.000 2.000
5 1.892 1.892 1.210 2.000 0.000 2.000 2.000 2.000 2.000 2.000
6 1.892 1.892 1.210 2.000 2.000 0.000 2.000 2.000 2.000 2.000
7 1.892 1.892 1.210 2.000 2.000 2.000 0.000 2.000 2.000 2.000
8 1.892 1.892 1.210 2.000 2.000 2.000 2.000 0.000 2.000 2.000
9 1.892 1.892 1.210 2.000 2.000 2.000 2.000 2.000 0.000 2.000
10 1.892 1.892 1.210 2.000 2.000 2.000 2.000 2.000 2.000 0.000
1 2 3 4 5 6 7 8 9 10
2
1.5
1
0.5
0



S_2
λ_2 = 1
Node
Node
1 2 3 4 5 6 7 8 9 10
0.000 2.000 2.000 2.000 2.000 2.000 2.000 2.000 2.000 2.000
2.000 0.000 2.000 2.000 2.000 2.000 2.000 2.000 2.000 2.000
2.000 2.000 0.000 2.000 2.000 2.000 2.000 2.000 2.000 2.000
2.000 2.000 2.000 0.000 2.000 2.000 2.000 2.000 2.000 2.000
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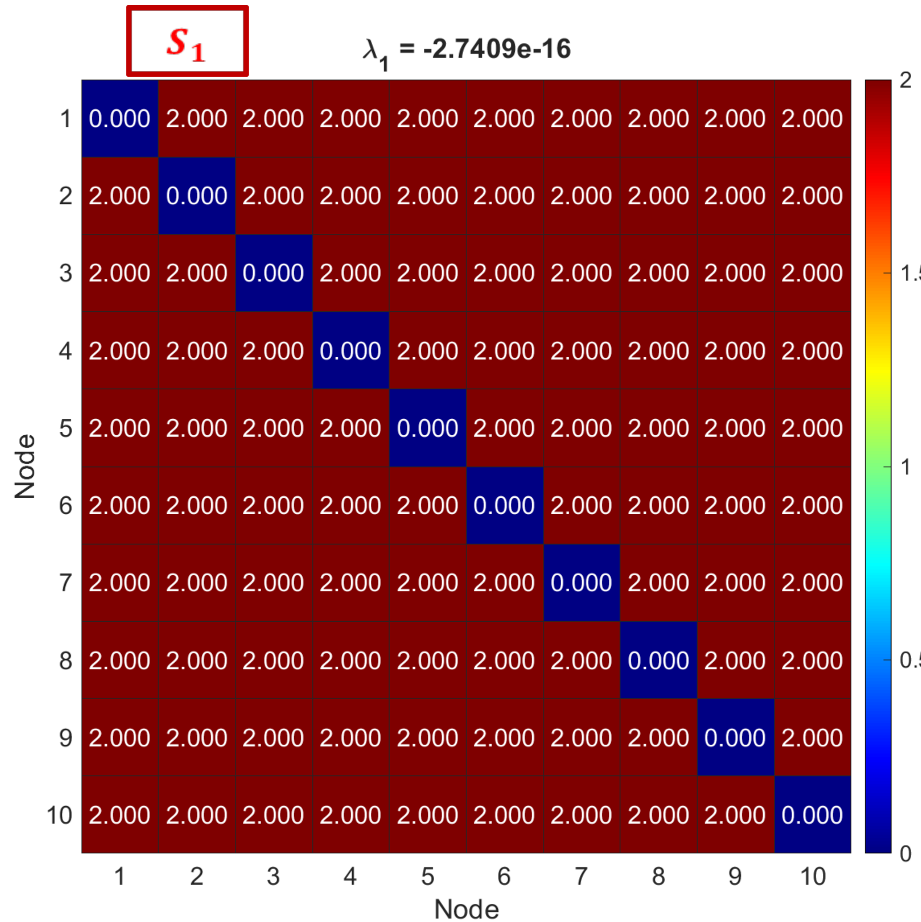
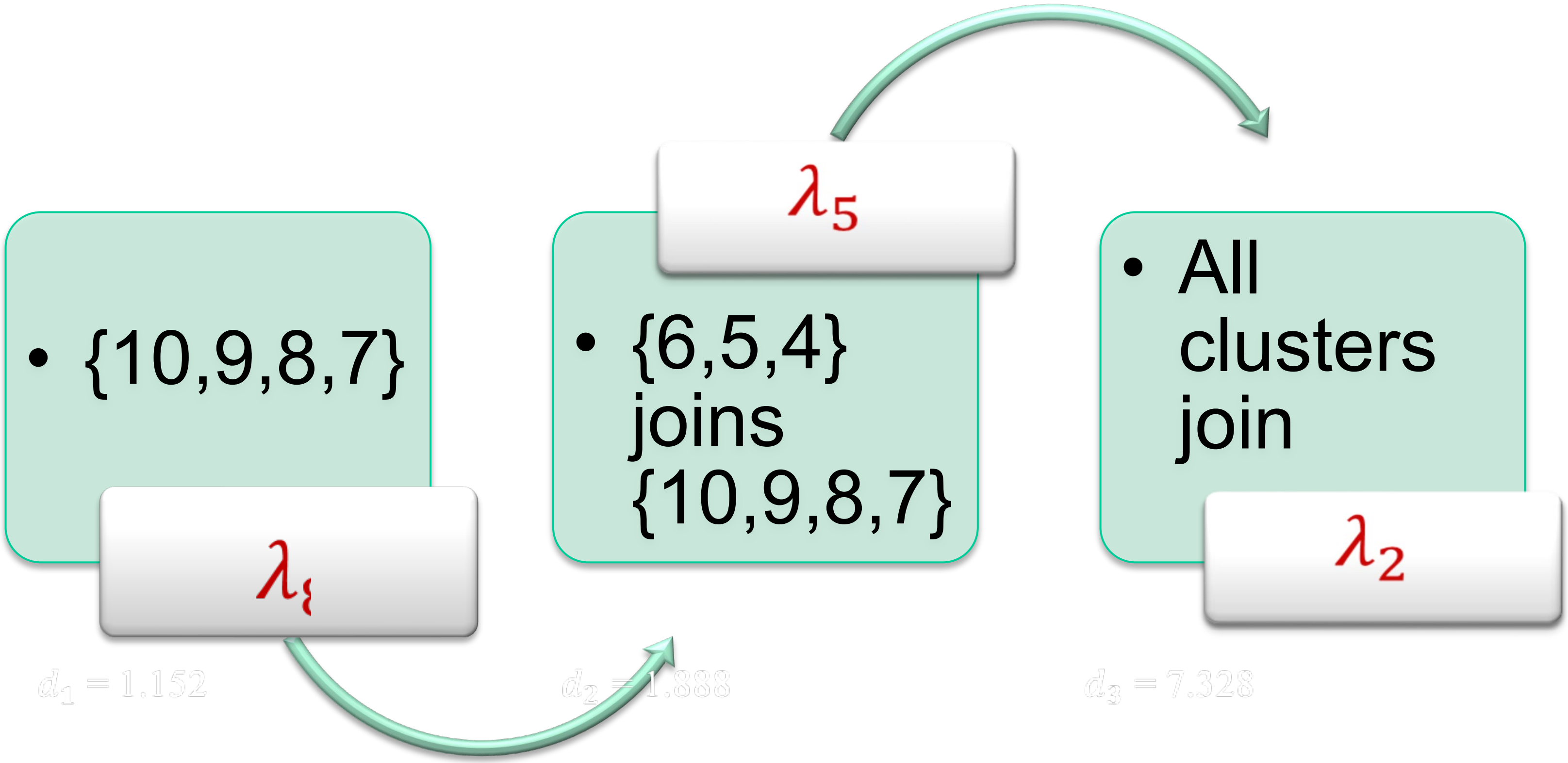
2.000 2.000 2.000 2.000 2.000 2.000 2.000 0.000 2.000 2.000
2.000 2.000 2.000 2.000 2.000 2.000 2.000 2.000 0.000 2.000
2.000 2.000 2.000 2.000 2.000 2.000 2.000 2.000 2.000 0.000
0 0.5 1 1.5 2

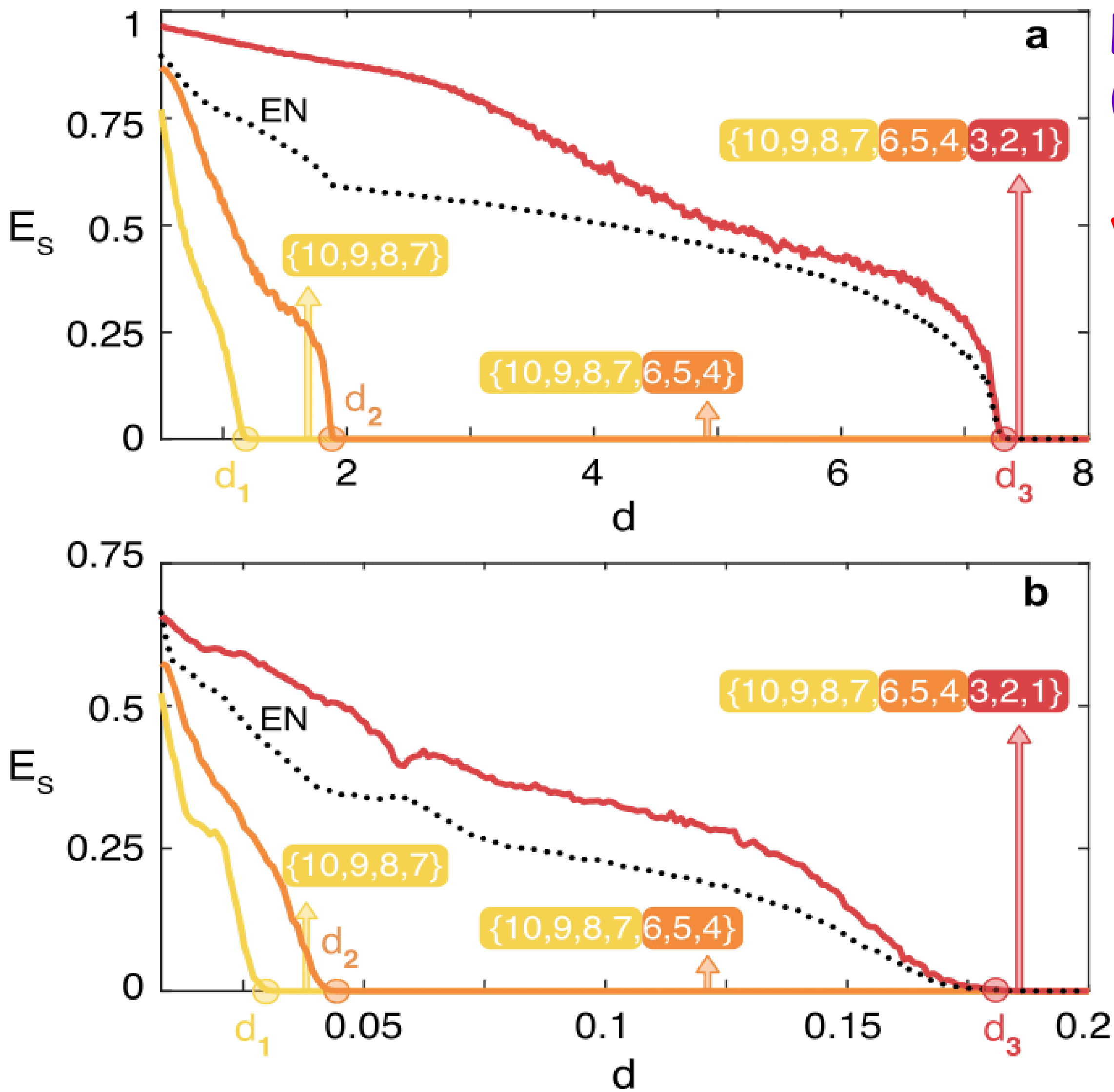


S_1
λ_1 = -2.7409e-16
Node
1 2 3 4 5 6 7 8 9 10
0.000 2.000 2.000 2.000 2.000 2.000 2.000 2.000 2.000 2.000
2.000 0.000 2.000 2.000 2.000 2.000 2.000 2.000 2.000 2.000
2.000 2.000 0.000 2.000 2.000 2.000 2.000 2.000 2.000 2.000
2.000 2.000 2.000 0.000 2.000 2.000 2.000 2.000 2.000 2.000
2.000 2.000 2.000 2.000 0.000 2.000 2.000 2.000 2.000 2.000
2.000 2.000 2.000 2.000 2.000 0.000 2.000 2.000 2.000 2.000
2.000 2.000 2.000 2.000 2.000 2.000 0.000 2.000 2.000 2.000
2.000 2.000 2.000 2.000 2.000 2.000 2.000 0.000 2.000 2.000
2.000 2.000 2.000 2.000 2.000 2.000 2.000 2.000 0.000 2.000
2.000 2.000 2.000 2.000 2.000 2.000 2.000 2.000 2.000 0.000
0 0.5 1 1.5 2

The predicted path to synchrony



The perfectly verified (and universal) path to synchrony!!!



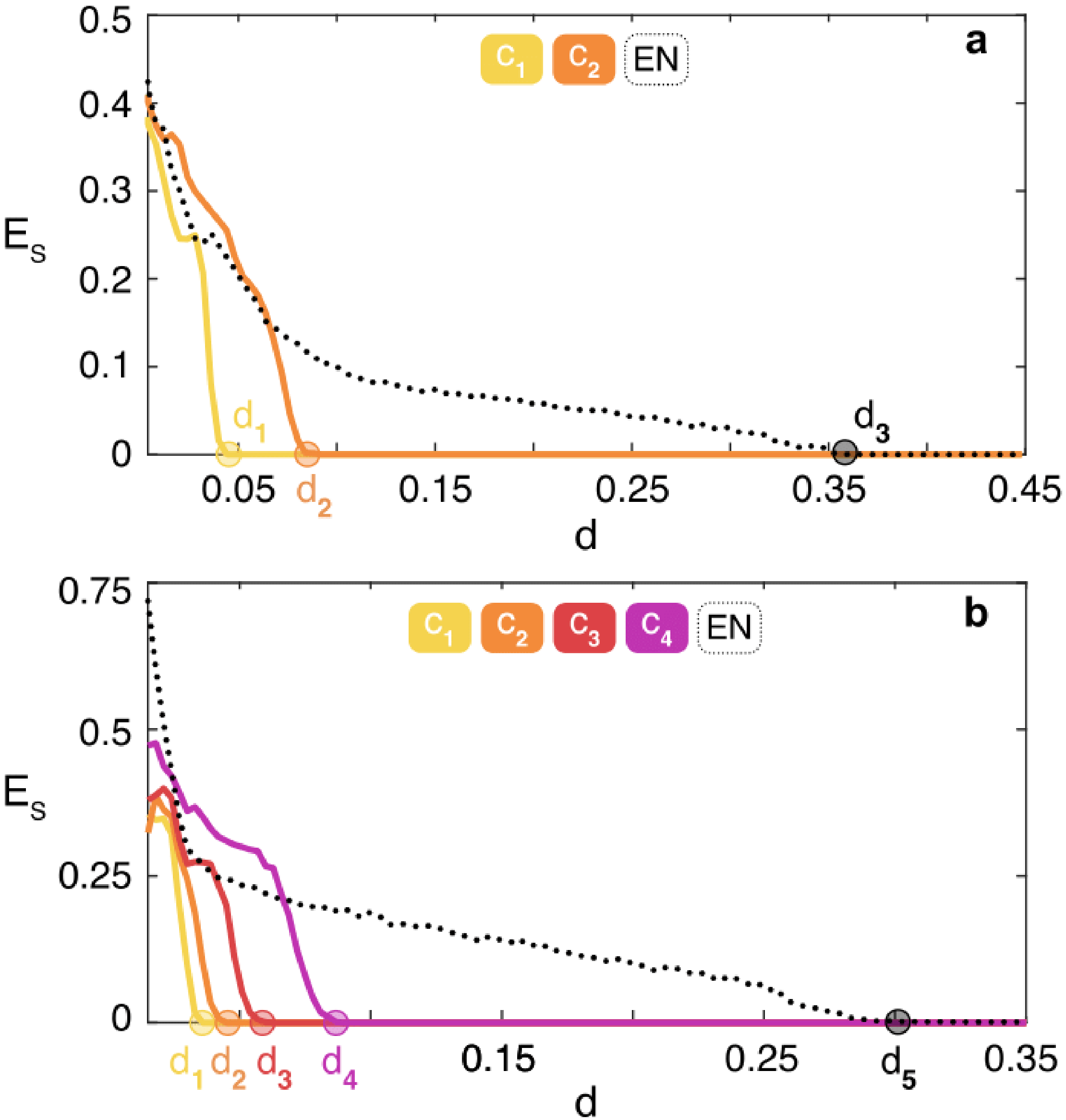
Lorenz system
(coupling on the x variable)

v*= 7.322

Roessler system
(coupling on the y variable)

v*= 0.179

Large size synthetic networks



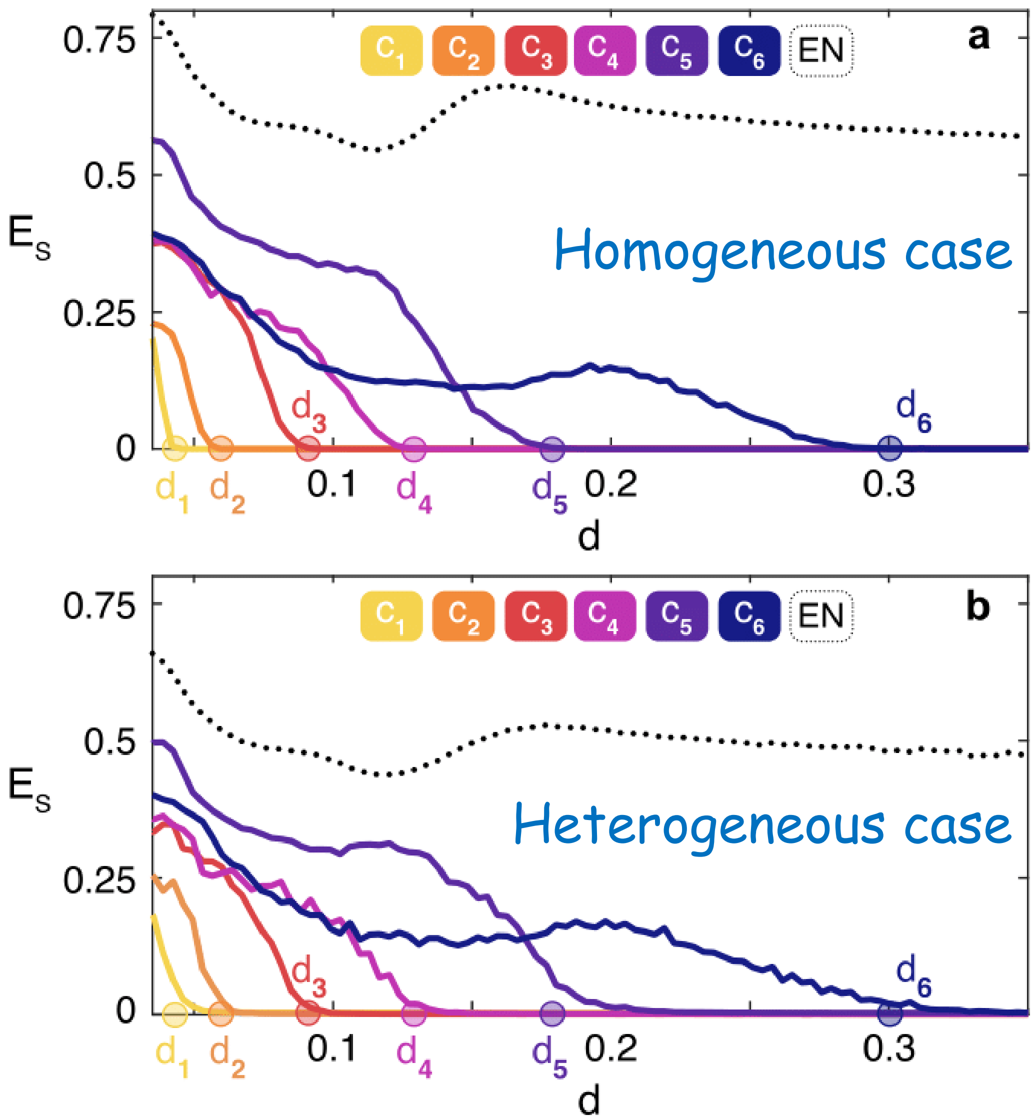
N=1,000

Two symmetry orbits leading to two distinct clusters:
20 nodes (Cluster 1)
10 nodes (Cluster 2).

N=10,000

Four symmetry orbits leading to four distinct clusters:
1,000 nodes (Cluster 1)
300 nodes (Cluster 2)
100 nodes (Cluster 3)
30 nodes (Cluster 4)

The PowerGrid network of the USA



N=4,941 with 6594 links

381 clusters found involving 871 nodes

We selected 6 clusters. The expected critical Values of d = ν*/λ are

$d_1 = 0.179 \times 0.25 = 0.04475$

$d_2 = 0.179 \times 0.333 = 0.0596$

$d_3 = 0.179 \times 0.5 = 0.0895$

$d_4 = 0.179 \times 0.723 = 0.1294$

$d_5 = 0.179 \times 1 = 0.179$

$d_6 = 0.179 \times 1.707 = 0.3056$

Thank you! (for your patience) and……….

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CHAOS, SOLITONS & FRACTALS

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