Politecnico di Torino

Information design in Bayesian routing games

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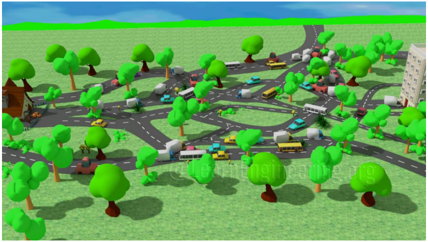
Linkoping, September 28th, 2023

Game theory

Game theory is the study of mathematical models of strategic interactions among rational agents[1]



[1] Myerson, Roger B. Game theory: analysis of conflict. Harvard university press, 1991

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- Mixed strategy NE is a probability distribution over the action set of each player such that no one wants unilaterally to deviate

Example: traffic lights

- Two players at a road intersection, $A_i = \{go, stop\}$ for each player i
  - Row = Player 1
  - Column = Player 2

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  - Equilibria are unfair
  - Total reward = 1
- One mixed-strategy NE: both players play $p = \begin{pmatrix} 1/101 \\ 100/101 \end{pmatrix}$
- The corresponding action distribution is pp', i.e.,

	go	*stop*
go	$\sim 0.01\%$	$\sim 1\%$
stop	$\sim 1\%$	$\sim 98\%$

  - Equilibrium is fair
  - Expected total reward = 0
- We would like equilibria to maximize the total reward and to be fair

Information in game theory: why we need mediators

- Rewards

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- Equilibria

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- This matrix is not in form pq'
- It is not feasible if actions are uncorrelated
- To correlate actions, we need a mediator!

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Take-home message

Information allows to correlate actions and induce desired outcomes

Information in traffic: motivation

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- Travel times may be affected by uncertainty (accidents, road works)
- Uncertainty may be leveraged by a mediator to reduce inefficiency

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- Given a flow f, the system cost is the expected travel time on the network

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- Flows depend on available information on the network state

Full information vs No Information

Full-info system optimum flow

- Full-info system-optimum is a flow that depends on θ and minimizes the system cost,

$$f_{FI}^* = \underset{f:\Theta\to\Delta_\epsilon}{\arg\min}\, C(f) = \underset{f:\Theta\to\Delta_\epsilon}{\arg\min} \int_\Theta \sum_{e\in\varepsilon} f_e(\theta)\tau_e(f_e(\theta),\theta)d\mathbb{P}(\theta).$$

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Full-info equilibrium flow

- System-optimum is not satisfying for users fully informed on θ
- Users take paths (links) with minimum delay. The equilibrium is $f^{W,FI}$ s.t.

$$f_e^{W,FI}(\theta) > 0 \implies \tau_e(f_e^{W,FI}(\theta),\theta) \leq \tau_i(f_i^{W,FI}(\theta),\theta) \quad \forall i \in \mathcal{E}$$

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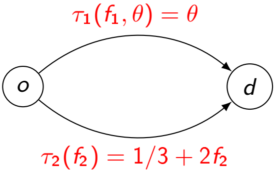
No-info equilibrium flow

- Users take links with minimum expected delay. The equilibrium is $f^{W,NI} \in \Delta_{\mathcal{E}}$ s.t.

$$f_e^{W,NI} > 0 \implies \mathbb{E}[\tau_e(f_e^{W,NI})] \le \mathbb{E}[\tau_i(f_i^{W,NI})] \quad \forall i \in \mathcal{E}$$

Example [Das17]



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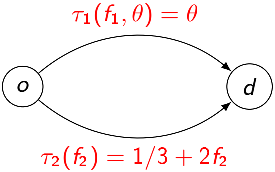

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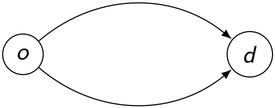

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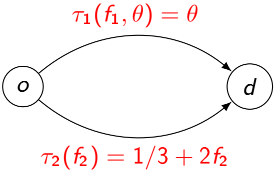

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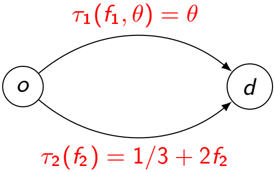

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Take-home message

- Providing information does not help the system at the equilibrium
- In general, information can also hurt the system [Acemoglu18]

Private signals

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- We consider private signals: send different signals to users
- Signals are route recommendations (revelation principle [Bergemann19])

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- Mediator observes state θ and sends signals according to π
- Users receive signal and choose route according to their posterior beliefs
- According to Bayes' theorem, the posterior belief of users that receive i is

$$d\mathbb{P}_i(\theta) = \frac{\pi_i(\theta) d\mathbb{P}(\theta)}{\int_\Theta \pi_i(\omega) d\mathbb{P}(\omega)}$$

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Definition (Bayesian user equilibrium)

- Let $\mathbb{E}_i[\cdot]$ denote expected value according to $d\mathbb{P}_i(\theta)$
- Given a policy π, a flow $f^{\pi,y}(\theta) = y'\pi(\theta)$ is a Bayesian user equilibrium if $\pi_i(\theta) y_{ij} > 0$ for at least a θ in Θ implies

$$\mathbb{E}_i[\tau_j(f^{\pi,y}(\theta), \theta)] \leq \mathbb{E}_i[\tau_k(f^{\pi,y}(\theta), \theta)] \quad \forall k \in \mathcal{E},$$

Proposition

Given a policy π, a flow $f^{\pi}(\theta) = (y^*)'\pi(\theta)$ is a Bayesian user equilibrium if and only if y^* is a solution of the convex program

$$y^* \in \underset{y \in \mathcal{R}_+^{\mathcal{E} \times \mathcal{E}} : y\mathbf{1} = \mathbf{1}}{\arg\min} \; \Phi(y, \pi),$$

where

$$\Phi(y, \pi) = \int_\Theta \sum_{e \in \mathcal{E}} \int_0^{(y'\pi(\theta))_e} \tau_e(\theta, s) ds d\mathbb{P}(\theta).$$

Equilibria characterization

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- The equilibrium f^{π} is unique for every policy π
- y^{*} equivalent to Nash equilibrium of a population game with weighted potential

- The cost of a policy π is the expected total travel time at the corresponding Bayesian user equilibrium f^π, i.e.,

$$C(\pi) = \int_\Theta \sum_e f_e^\pi(\theta)\tau_e\left(f_e^\pi(\theta), \theta\right) d\mathbb{P}(\theta).$$

Problem formulation

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Problem

The system planner wants to find π^* such that

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- It is a bi-level program, since $f^{\pi} = (y^*)'\pi$ and $y^* = \arg\min \Phi\left(y, \pi\right)$
- Can we simplify the problem?

Obedience and revelation principle

Definition: obedience

A policy π is obedient if users do not want to deviate, i.e., if $y^* = I$

$$I \in \underset{y \in \mathcal{R}_+^{\mathcal{E}\times\mathcal{E}} : y\mathbf{1}=\mathbf{1}}{\arg\min} \Phi\left(y, \pi\right) = \int_\Theta \sum_{e\in\mathcal{E}} \int_0^{(y'\pi(\theta))_e} \tau_e(\theta, s) ds d\mathbb{P}(\theta).$$

In other words, π is obedient if $f^\pi = \pi$

Revelation principle [Bergemann19]

- For every policy π, there always exists an obedient policy $\tilde{\pi}$ such that $C(\pi) = C(\tilde{\pi})$
- Revelation principle implies that we can restrict attention to obedient policies

Problem formulation II

- Revelation principle: restrict to obedient policies ($f^\pi = \pi$)

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The optimal information policy is

$$\pi^* = \underset{\pi:\Theta\to\Delta_\mathcal{E}}{\arg\min} \int_\Theta \sum_{e\in\mathcal{E}} f_e^\pi(\theta)\tau_e(f_e^\pi(\theta),\theta)d\mathbb{P}(\theta)$$

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$$f^\pi = (y^*)'\pi, \quad y^* = \arg\min_y \Phi(y,\pi)$$

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subject to

$$\mathbb{E}_i[\tau_i(\pi_i(\theta),\theta) - \tau_j(\pi_j(\theta),\theta)] \leq 0, \quad \forall i,j \in \mathcal{E}$$

- The problem is now single-level ($C(\pi)$ depends directly on π)
- The price is that we now have obedience constraints (# constraints = $|\mathcal{E}|(|\mathcal{E}|-1)$)

Some literature

- [Das17] raises the problem and analyzes examples
- [Tafavoghi17] provides sufficient conditions for optimality in a stylized setting
- [Savla22] focuses on computational aspects

Problem properties

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Properties

- The objective function is always convex in π
- Obedience constraints are in general non-convex
- Obedience constraints are convex with two parallel links and affine delay functions

Price of Anarchy

Definition

The Price of Anarchy (PoA) is

$$PoA = \frac{C(\pi^*)}{C(f^*)} = \frac{\int_\Theta \sum_{e \in \mathcal{E}} \pi_e^*(\theta)\tau_e(\pi_e^*(\theta), \theta) d\mathbb{P}(\theta)}{\int_\Theta \sum_{e \in \mathcal{E}} f_e^*(\theta)\tau_e(f_e^*(\theta), \theta) d\mathbb{P}(\theta)}$$

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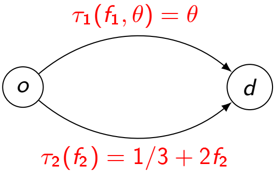
- $PoA \geq 1$ by construction
- $PoA = 1$ iff $\pi^* = f^*$
- Is non-convexity a problem? It depends!
- If we search conditions under which optimality can be achieved (PoA=1), no!
  - Compute $f^*(\theta)$ for every θ (minimize convex function)
  - PoA=1 iff f^* satisfies obedience constraints
- Finding π^* when $PoA > 1$ may be hard because of non-convexity.

Back to example: can information achieve optimality?



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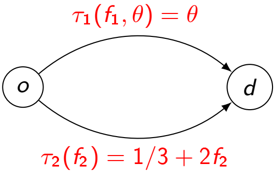

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- Full-info system optimum: $f^*(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, f^*(1) = \begin{pmatrix} 5/6 \\ 1/6 \end{pmatrix}, C(f^*) = 17/36$

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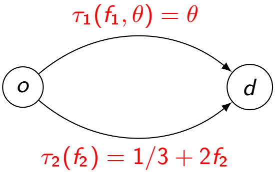

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- If an agent receives signal $s = 1$,

$$\mathbb{E}_1[\tau_1] = \mathbb{P}(\theta = 1 | s = 1) = \frac{\mathbb{P}(\theta = 1)\overbrace{\mathbb{P}(s = 1 | \theta = 1)}^{=\pi_1(1)}}{\mathbb{P}(s = 1)} = \frac{\frac{1}{2} \times \frac{5}{6}}{\frac{1}{2} \times 1 + \frac{1}{2} \times \frac{5}{6}} = \frac{5}{11}$$

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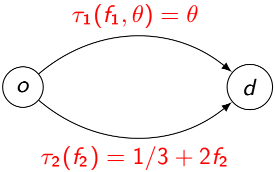

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$$\mathbb{E}_1[\tau_2] = \frac{1}{3} + 2\left(\mathbb{P}(\theta = 0 | s = 1)\pi_2(0) + \mathbb{P}(\theta = 1 | s = 1)\pi_2(1)\right) = \frac{1}{3} + \frac{5}{33} = \frac{16}{33} > \frac{5}{11}$$

- Since $\mathbb{E}_1[\tau_1] \leq \mathbb{E}_1[\tau_2]$ (same for link 2), f^* is obedient, hence $\pi^* = f^*$ and $PoA = 1$.

Two links, affine delay functions

- Two links, delay functions

$$\tau_e(a_e, b_e, f_e) = a_e f_e + b_e, \quad e = 1, 2$$

- a number (for ease of exposition), b random variable

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Problem

Find $\pi_1^* : \Theta \to [0, 1]$ to minimize

$$\int_\Theta [(x - 2a_2)\pi_1(a, x) + (a_1 + a_2)\pi_1^2(a, x)] d\mathbb{P}(a, x)$$

under obedience constraints

$$\int_\Theta [(x - a_2)\pi_1(a, x) + (a_1 + a_2)\pi_1^2(a, x)] d\mathbb{P}(a, x) \leq 0$$

$$\int_\Theta [a_2 - x + (x - a_1 - 2a_2)\pi_1(a, x) + (a_1 + a_2)\pi_1^2(a, x)] d\mathbb{P}(a, x) \leq 0$$

Results: conditions for optimality

- Full-info system optimum is

$$f_1^* (a, x) = \left[\frac{2a_2 - x}{2 (a_1 + a_2)} \right]_0^1$$

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Theorem 1 [Cianfanelli23]

Assume that

$$supp(x) \subseteq [-2a_1, 2a_2]. \quad (1)$$

Then, $\pi^* = f^*$ and $PoA = 1$ if and only if

$$\begin{cases} \sigma_x^2 \geq \mathbb{E}[x](2a_2 - \mathbb{E}[x]) & \text{(obedience 1 vs 2)} \\ \sigma_x^2 \geq -\mathbb{E}[x](2a_1 + \mathbb{E}[x]) & \text{(obedience 2 vs 1)} \end{cases} \quad (2)$$

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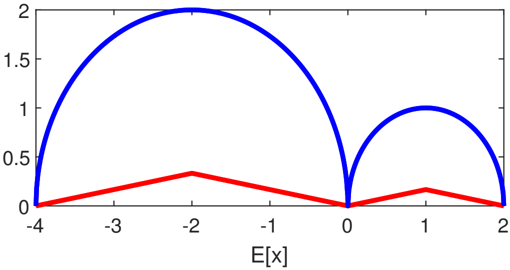
- (1) guarantees that f^* does not saturate
- (2) is verified iff f^* satisfies obedience constraints
- Remark: (1) is not necessary

How much uncertainty do we need?

Example: $a_1 = 2, a_2 = 1$



- Smallest $\sqrt{\sigma_x^2}$ such that $PoA = 1$ (optimality)
- $||f^W - f^*||_1$ in no-information setting as a function of $\mathbb{E}[x]$

- Theorem 1 shows that if σ_x^2 is large, optimality can be achieved more easily
- The mediator leverages the uncertainty to persuade the agents
- Remark 1: Minimum variance σ_x^2 for optimality depends on distance between equilibrium and system-optimum
- Remark 2: if $\mathbb{E}[x] = 0$, no uncertainty needed

What if assumptions are not met?

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Other results

- When (4) violated, we can compute π^* and PoA to measure suboptimality
- We have studied a special case with b_e uniform in $[0, 1]$ (hence $\mathbb{E}[x] = 0$), showing that $PoA = 1$ for every a, even if (3) is violated

N links

- N parallel links, delay functions

$$\tau_e(a_e, b_e, f_e) = a_e f_e + b_e, \quad e = 1, \cdots, N$$

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Theorem 2 (optimality result)

Assume that

$$\min_{e \in \mathcal{E}} \min_{b} \sum_{i \in \mathcal{E}} \frac{b_i - b_e}{2a_i} \geq -1 \tag{5}$$

Then, $PoA = 1$ if and only if

$$\mathbb{E}\left[\left(2\prod_{k \neq j} a_k + \sum_{i}(b_i - b_j)\prod_{k \neq i,j} a_k\right)(b_j - b_e)\right] \leq 0\,, \quad \forall j, e \in \mathcal{E} \tag{6}$$

- Not much intuition, but...

N links

Proposition 1

Assume that

$$\mathbb{E}[b_e] = c\,, \quad \forall e \in \mathcal{E} \tag{7}$$

$$\mathbb{E}[b_i b_j] - \mathbb{E}[b_i]\mathbb{E}[b_j] := K_{ij} = 0\,, \quad \forall i \neq j. \tag{8}$$

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- (7) generalizes $\mathbb{E}[x] = 0$ (recall $x = b_1 - b_2$) to the case of N links
- (8) was not required with 2 links
- Conjecture: Proposition 1 holds true for arbitrary topologies, if all the paths have same expected free-flow delay

The role of correlations: 3 parallel links

- 3 parallel links with $\tau_e = a_e f_e + b_e$
- a_e is a number, $\mathbb{E}[b_e] = c$ for every link e
- There are 6 obedience constraints
- We focus on "*who gets signal* 1 *(with policy* $\pi = f^*$*) prefers* 1 *over* 2"

$$-K_{11}(a_2 + a_3) - K_{22}a_3 + (a_2 + 2a_3)K_{12} + a_2 K_{13} - K_{23}a_2 \leq 0.$$

- Off-diagonal elements ok covariance K play a role

Example

Assume that b_3 is known ($K_{i3} = 0$ for every i), $a_2 = a_3 = 1$. Then, the constraint becomes

$$K_{12} \leq \frac{2K_{11} + K_{22}}{3}.$$

Let $K_{11} = 1, K_{22} = 2$. Then, if $K_{12} > 4/3$, the obedience constraint is violated.

Remark: With 2 links, $\mathbb{E}[x] = \mathbb{E}[b_1 - b_2] = 0$ guarantees optimality, regardless $K_{12} = 0$. With 3 links, this is no longer sufficient (neither necessary)

Conclusions

Summary

- Formulated information design problem in routing games
- Sufficient conditions for optimality with N parallel links and affine delays
- Correlations can hurt the system

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Future research lines

- Extend results to general topologies and delay functions
- Consider information design in other games
- Study what happens when multiple routing apps compete for customers