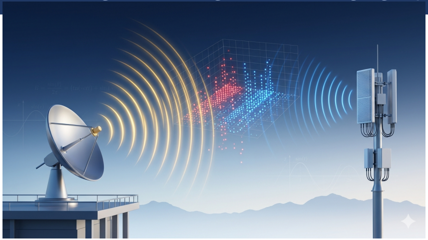
Extremal Real Entire Functions of Exponential Type for Blind Spectral Coexistence

Elliit Focus Period 2026: Wireless Sensing Technologies for Emerging Applications



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Motivation

Motivation

IEEE Radar band	VHF/UHF [30 MHz – 1 GHz]	L [1-2 GHz]	S [2-4 GHz]	C [4-8 GHz]	X [8-12 GHz]	Ku, K, Ka ,V, W [12-300 GHz]
Examples of radar usage	FOPEN	ARSR	ASR, NEXRAD	TDWR	CASA	Automotive radars, cloud radars
Co-existing comms	TV/broadcast/802.11a h/f	WiMAX, JTIDS	LTE	802.11a/ac	LTE	802.11ad, mmwave comm

Shared spectrum is here to stay

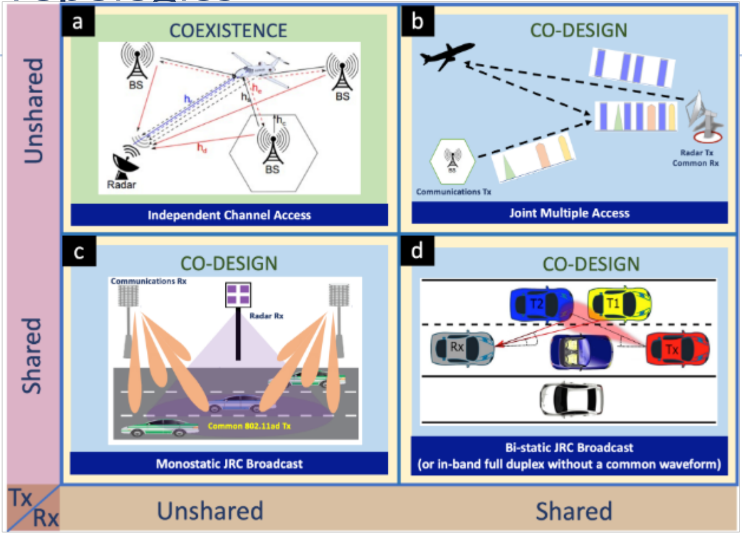
Radar and wireless systems (5G, Wi–Fi, automotive) are being pushed into *the same* frequency bands. The receiver sees a **superposition** of echoes and messages.

The blind regime

- radar **waveform** is unknown ,
- communications **message** is unknown,
- **channels** (delays, Dopplers, DoAs) are unknown.

$\Rightarrow$ severely ill–posed unless we exploit structure.

ISAC Topologies



K. V. Mishra *et al.*, "Toward mmWave Joint Radar Communications: A Signal Processing Perspective," *IEEE*

Blind Co-design Receiver

What the receiver observes

$$y(t) = \underbrace{x_r(t) * h_r(t)}_{\text{radar trail}} + \underbrace{x_c(t) * h_c(t)}_{\text{comms trail}} + \xi(t)$$

with, after processing, the discrete model

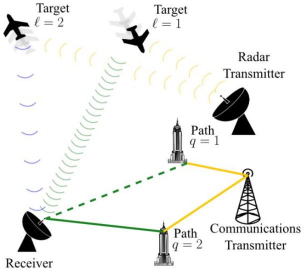
$$[\mathbf{y}]_{\tilde{n}} = \sum_{\ell} [\boldsymbol{\beta}]_{\ell} [\mathbf{s}]_n e^{-j2\pi(n\Delta f[\boldsymbol{\tau}_r]_{\ell} + p[\boldsymbol{\nu}_r]_{\ell})}$$

$$+ \sum_{q} [\boldsymbol{\omega}]_q [\mathbf{g}]_{\tilde{n}} e^{-j2\pi(n\Delta f[\boldsymbol{\tau}_c]_q + p[\boldsymbol{\nu}_c]_q)} + [\boldsymbol{\xi}]_{\tilde{n}}$$

Unknowns

- Radar: $\{\boldsymbol{\tau}_r, \boldsymbol{\nu}_r, \boldsymbol{\beta}\}$ – L targets
- Comms: $\{\boldsymbol{\tau}_c, \boldsymbol{\nu}_c, \boldsymbol{\omega}\}$ – Q paths
- Waveforms: s (radar), g (OFDM symbols)

Recover $2(L+Q)$ continuous parameters **plus** waveforms from MP noisy samples.



Prior Art & Our Contributions

Prior dual-blind deconvolution (DBD) approaches

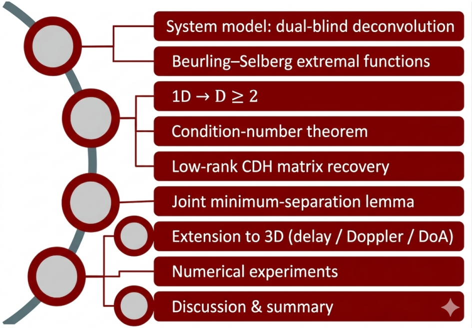
- **SoMAN minimization** (Vargas *et al.*, J-SAIT 2023): sum of multivariate atomic norms; SDP relaxation; separation requirements *independent* for radar and comms: $\min(\delta_{\tau,r}, \delta_{\nu,r}, \delta_{\tau,c}, \delta_{\nu,c}) \geq c/\min(M, P)$.
- **Factor-graph EM** (Jacome *et al.*, CAMSAP 2023): tracks a linear state-space model; Gaussian assumption; no Doppler.

This work

1. A **joint separation** condition δ in the delay–Doppler plane, with $\delta \geq O(1/\sqrt{MP})$ instead of $O(1/\min(M,P))$.
2. A **unified** nuclear-norm recovery of a single low-rank *concatenated D-fold Hankel* (CDH) matrix.
3. **Multidimensional** Beurling–Selberg extremal functions $\Rightarrow$ bound on $\kappa(\mathbf{W})$ for any $D \geq 2$.
4. **Extension to 3D** (delay, Doppler, DoA) with ULA receivers.

Outline

- System model: dual-blind deconvolution
- Beurling–Selberg extremal functions
  $1D \to D \geq 2$
- Condition-number theorem
- Low-rank CDH matrix recovery
- Joint minimum-separation lemma
- Extension to 3D (delay / Doppler / DoA)
- Numerical experiments
- Discussion & summary



Dual-Blind Deconvolution: System Model

Transmit Model

Radar (Pulse–Doppler, Swerling-I)

$$x_r(t) = \sum_{p=0}^{P-1} s(t - pT), \qquad 0 \le t \le PT$$

Channel with L targets:

$$h_r(t,\tau) = \sum_{\ell=0}^{L-1} [\boldsymbol{\beta}]_\ell \delta(\tau - [\boldsymbol{\tau}_r]_\ell)\, e^{-j2\pi[\boldsymbol{\nu}_r]_\ell t}$$

Communications (OFDM)

$$x_c(t) = \sum_{m=0}^{P-1} x_m(t-mT), x_m(t) = \sum_{k=0}^{K-1} [\mathbf{g}_m]_k e^{j2\pi k \Delta f t}$$

Channel with Q paths:

$$h_c(t,\tau) = \sum_{q=0}^{Q-1} [\boldsymbol{\omega}]_q \delta(\tau - [\boldsymbol{\tau}_c]_q)\, e^{-j2\pi[\boldsymbol{\nu}_c]_q t}$$

Low-rank structural assumption

Waveform and messages live in a *low-dimensional* subspace: $\mathbf{s} = \mathbf{B}\mathbf{u}, \quad \mathbf{g}_p = \mathbf{D}_p \mathbf{v}_p, \quad \mathbf{B}, \mathbf{D}_p \in \mathbb{C}^{M \times J}, \; J \ll M.$

Concatenated Vandermonde–Khatri–Rao Model

After DFT along range bins and stacking P pulses,

Linear measurement model

$$\boxed{\mathbf{y} = \mathbf{W}(\boldsymbol{\tau}, \boldsymbol{\nu})\,\boldsymbol{\eta} + \boldsymbol{\xi}} \qquad \mathbf{W} \in \mathbb{C}^{MP\times J(L+Q)}$$

$$\mathbf{W} = [\mathbf{W}_r, \mathbf{W}_c], \qquad \boldsymbol{\eta} = [\,[\beta]_0\mathbf{u}^T, \ldots, [\omega]_{Q-1}\mathbf{v}^T\,]^T$$

Why the condition number of $\mathbf{W}$ matters

- Noise stability of waveform recovery $\mathrm{minimize}_{\boldsymbol{\eta}}\, \|\mathbf{y} - \mathbf{W}\boldsymbol{\eta}\|_2^2$ is governed by $\kappa(\mathbf{W}) = \sigma_{\max}/\sigma_{\min}$.
- $\mathbf{W}$ has a *Vandermonde–Khatri–Rao* structure; its columns are entry-wise weighted 2D Vandermonde vectors.
- We need a **separation condition** on $(\boldsymbol{\tau}, \boldsymbol{\nu})$ that guarantees $\mathbf{W}$ is well conditioned.

Beurling–Selberg Extremal Functions

Extremal Real Entire Functions, $D = 1$

Definition

$F : \mathbb{C} \to \mathbb{C}$ is a **real entire function of exponential type** 2π iff

C1 F is entire,

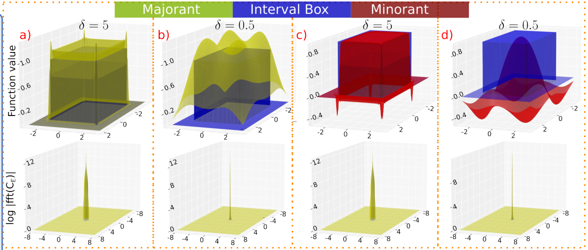
C2 $F|_{\mathbb{R}}$ is real-valued,

C3 $|F(z)| \le C\, e^{(2\pi+\varepsilon)|z|}$ for every $\varepsilon > 0$.

Beurling–Selberg bound

For the interval $E = [a, b]$, $I_E(t) = \mathbf{1}_{[a,b]}(t)$, there exist entire functions $c_E(\delta t) \le I_E(t) \le C_E(\delta t)$ with:

- Fourier support in $[-\delta, \delta]$,
- $\int (C_E - I_E)\, dt = 1/\delta = \int (I_E - c_E)\, dt$,
- *extremal:* minimise the error integral.



Other Approximations to $I_E(t)$

Approach	**Expression**	**Property**	$D \geq 2$**?**
Beurling–Selberg	$C_E(t) = \frac{1}{2}\{B^+(t-a) + B^+(b-t)\}$	$C_E \geq I_E$, extremal	Yes
Babenko	Zeros of Bernstein $B(t)$	Approximation on $[-\pi, \pi]$	No
Stenger	$S(t) = \frac{1}{1+k}[1 + k\phi(t)]$	Analytic, $(-1, 1)$ only	No

Why we pick Beurling–Selberg

Only BS functions (i) **bound** I_E from both sides, (ii) have **compact Fourier support**, and (iii) **generalise** to $D \geq 2$ while keeping the two-sided bound – exactly what we need to bound $\kappa(\mathbf{W})$.

Generalisation to $D \geq 2$

Paley–Wiener (higher dimension)

$f(\mathbf{z})$ entire on $\mathbb{C}^D$ is the FT of a function supported in the ball $\{|\mathbf{z}| \leq R\}$ iff for every N, $|f(\mathbf{z})| \leq \dfrac{C_N e^{R|\mathbf{z}|}}{(1+|\mathbf{z}|)^N}$.

Majorant and (Todd) minorant on a box $E = \prod_d [a_d, b_d]$

$$C_E(\mathbf{t}) = \prod_{d=1}^{D} C_E(t_d), \qquad c_E^3(\mathbf{t}) \text{ (Todd)}: \quad I_E(\mathbf{t}) \geq c_E^3(\mathbf{t}).$$

FT of $C_E(\boldsymbol{\delta}\mathbf{t})$, $c_E^3(\boldsymbol{\delta}\mathbf{t})$ supported in $\prod_d [-\delta_d, \delta_d]$. With $\boldsymbol{\delta} = \delta \mathbf{1}$ the error integrals become

$$\int (C_E - I_E)\, d\mathbf{t} = \int (I_E - c_E^3)\, d\mathbf{t} = \prod_d \left(b_d - a_d + \tfrac{1}{\delta}\right) - \prod_d (b_d - a_d).$$

Key consequence

Plugging this error integral into the 2D (and 3D) receive-signal energy bound produces a *non-trivial* range of delay–Doppler (and DoA) separations for which $\mathbf{W}$ is well conditioned.

Condition Number Bound

Joint Minimum Separation

Wrap-around distance

$$d(a,b) = \min(|a-b|,\, 1-|a-b|).$$

Why "joint"?

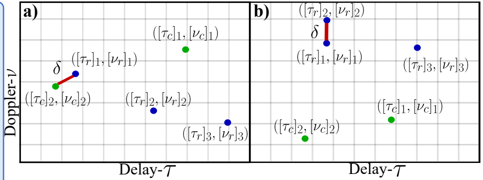
Two entries with *identical delay* can still be recovered if their Dopplers differ – and vice-versa. SoMAN cannot handle this.

Joint separation

With $\boldsymbol{\tau} = [\boldsymbol{\tau}_r^T, \boldsymbol{\tau}_c^T]^T$, $\boldsymbol{\nu} = [\boldsymbol{\nu}_r^T, \boldsymbol{\nu}_c^T]^T$,

$$\delta = \min_{i \neq j} \psi\big(d([\boldsymbol{\tau}]_i, [\boldsymbol{\tau}]_j),\, d([\boldsymbol{\nu}]_i, [\boldsymbol{\nu}]_j)\big)$$

where $\psi(x,y) = \max(x,y)$ or $\sqrt{x^2+y^2}$.



Main Theorem – Condition Number Bound

Theorem (Monsalve, Vargas, Mishra, Sadler, 2025)

Let $\gamma(\delta) = \delta P + \delta(M-1) + 1$. Then

$$\boxed{\kappa^2(\mathbf{W}) \;\leq\; \frac{MP + \dfrac{\gamma(\delta)}{\delta^2} - P}{MP - \dfrac{\gamma(\delta)}{\delta^2} - P}}$$

Non-triviality ($\mathbf{W}$ full column rank) requires

$$M > \frac{\delta P(\delta+1) - \delta + 1}{\delta(\delta P - 1)}, \qquad P > \frac{1}{\delta}.$$

Taking $P = c/\delta$ with $c > 1$ yields $M > (3+\delta)/\delta$ for $c = 2$.

Comparison with SoMAN (Vargas *et al.* 2023)

- SoMAN: $\min(\delta_\tau, \delta_\nu) \geq O(1/\min(M, P))$.
- Here: $\delta > O(1/\sqrt{MP})$.

Proof Sketch

Idea (delay–Doppler case)

Sample with a 2D Dirac comb $h(n,p) = \sum_{i,j} \delta(n-i, p-j)$ and bound $\|\mathbf{y}\|^2$ above/below by $\int h(n,p)\, C_E(n,p)\, |Y(n,p)|^2\, dn\, dp$ and the analogous minorant integral.

Key step

Expand $Y(n,p)$: cross terms involve $\alpha^{([\boldsymbol{\tau}_r]_\ell - [\boldsymbol{\tau}_c]_q)n + ([\boldsymbol{\nu}_r]_\ell - [\boldsymbol{\nu}_c]_q)p}$. Because $\widetilde{C}_E$ is supported in $(-\delta,\delta)^2$ and any two distinct atoms are *jointly* δ-apart, all cross-terms vanish. Only the diagonal survives, and

$$\widetilde{C}_E(\mathbf{0}) = MP + \frac{\gamma(\delta)}{\delta^2} - P, \qquad \widetilde{c}_E(\mathbf{0}) = MP - \frac{\gamma(\delta)}{\delta^2} - P.$$

Conclusion

Sandwiching $\|\mathbf{y}\|^2$ between these constants times a positive γ factor yields the κ^2 bound because $\kappa(\mathbf{W}) = \max \|\mathbf{W}\boldsymbol{\lambda}\| / \min \|\mathbf{W}\boldsymbol{\lambda}'\|$.

Low-Rank CDH Matrix Recovery

From Measurements to a Low-Rank Matrix

Hidden data matrices

Stack radar/comms unknowns pulse-by-pulse:

$$\mathbf{X}_r = [\mathbf{u}\mathbf{h}_{r,0}^T, \ldots, \mathbf{u}\mathbf{h}_{r,P-1}^T], \qquad \mathbf{X}_c = [\mathbf{v}_0\mathbf{h}_{c,0}^T, \ldots, \mathbf{v}_{P-1}\mathbf{h}_{c,P-1}^T]$$

with $\mathbf{h}_{r,p} = \sum_\ell [\boldsymbol{\beta}]_\ell \mathbf{a}([\boldsymbol{\tau}_r]_\ell) e^{-2j\pi[\boldsymbol{\nu}_r]_p}$ and analogous for comms. Then $\mathbf{y} = \mathcal{A}(\mathbf{X})$, $\mathbf{X} = [\mathbf{X}_r^T, \mathbf{X}_c^T]^T$.

Two-fold vectorised Hankel lift

Choose $N_1 + N_2 = M + 1$, $P_1 + P_2 = P + 1$. Define

$$\mathcal{H}_2(\mathbf{Z}) = \begin{bmatrix} \mathcal{H}(\mathbf{Z}_0) & \cdots & \mathcal{H}(\mathbf{Z}_{P_2-1}) \\ \vdots & \ddots & \vdots \\ \mathcal{H}(\mathbf{Z}_{P_1-1}) & \cdots & \mathcal{H}(\mathbf{Z}_{P-1}) \end{bmatrix}$$

and the Concatenated D-fold Hankel (CDH) operator

Factorisation and Nuclear-Norm Recovery

Low-rank factorisation

$$\mathcal{C}(\mathbf{X}) = \mathbf{L}\,\mathrm{diag}(\boldsymbol{\psi})\,\mathbf{R}^T, \qquad \mathrm{rank}(\mathcal{C}(\mathbf{X})) \leq L + P_2 Q$$

with explicit $\mathbf{L} \in \mathbb{C}^{JP_1N_1 \times L+P_2Q}$ and $\mathbf{R} \in \mathbb{C}^{2N_2P_2 \times L+P_2Q}$ built from Vandermonde blocks and waveform/message subspaces.

Proposed convex recovery

$$\underset{\mathbf{X}}{\text{minimize}} \;\; \|\mathcal{C}(\mathbf{X})\|_* \qquad \text{s.t.} \qquad \mathbf{y} = \mathcal{A}(\mathbf{X})$$

then:

- MUSIC / ESPRIT on $\mathcal{H}_2(\mathbf{X}_r)^T$, $\mathcal{H}_2(\mathbf{X}_c)^T \Rightarrow (\boldsymbol{\tau}_r, \boldsymbol{\nu}_r)$, $(\boldsymbol{\tau}_c, \boldsymbol{\nu}_c)$;
- Form $\mathbf{W}$ and solve $\min_{\boldsymbol{\eta}} \|\mathbf{y} - \mathbf{W}\boldsymbol{\eta}\|_2 \Rightarrow$ waveforms.

Recovery Guarantee

Assumptions on B and D (columns i.i.d.)

P1 Isotropy: $\mathbb{E}[\mathbf{b}\mathbf{b}^*] = \mathbf{I}$.

P2 Incoherence: $\max_j |[\mathbf{b}]_j| \leq \mu_0$.

P3 Lower boundedness: $\|\mathbf{b}\|_2^2 \geq 1$.

P4 Uncorrelated messages: $\mathbb{E}[\mathbf{v}_i^* \mathbf{v}_j] = 0,\ i \neq j$.

Theorem (Chen *et al.* 2022, applied here)

If $\sigma_{\min}(\mathbf{L}^*\mathbf{L}) \geq N_1 P_1/\mu_1$, $\sigma_{\min}(\mathbf{R}^*\mathbf{R}) \geq N_2 P_2/\mu_1$ with $\mu_1 > 1$, and $MP \geq \mu_0 \mu_1 J(L+Q)\log^5(JMP)$, then nuclear-norm minimisation recovers $\mathcal{C}(\mathbf{X})$ with probability $\geq 1 - c_0 (JMP)^{-c_1}$.

Lemma 8 (Monsalve *et al.* 2025)

The singular-value conditions above hold provided

$$\boxed{\delta \geq \frac{\mu_1(2P+2M)}{(\mu_1-1)(M+1)(P+1)}}$$

which is $O(1/\min(M,P))$ in the worst case and $O(1/\sqrt{MP})$ for balanced regimes.

Extension to 3D: Delay / Doppler / DoA

ULA Receiver: Adding Direction of Arrival

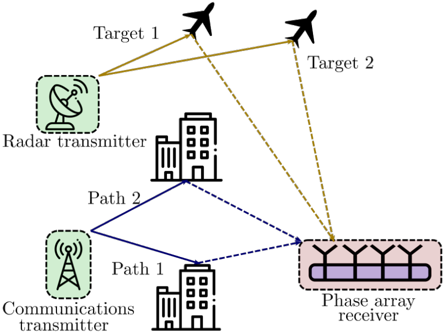
Signal at antenna $\eta \in \{0, \dots, N_a - 1\}$

$$y(t, \eta, p) = \sum_{\ell} [\boldsymbol{\beta}]_\ell s(t - [\boldsymbol{\tau}_r]_\ell) e^{-j2\pi[\boldsymbol{\nu}_r]_\ell t} e^{-j2\pi[\boldsymbol{\omega}_r]_\ell \eta}$$

$$+ \text{ comms term with } [\boldsymbol{\omega}_c]_q,$$

where $[\boldsymbol{\omega}_r]_\ell = \sin([\boldsymbol{\theta}_r]_\ell / 2)$. The 3D Vandermonde matrix:

$$\mathbf{V}_r = \mathbf{a}_{N_a}(\boldsymbol{\omega}_r) \odot \mathbf{a}_P(\boldsymbol{\nu}_r) \odot \mathbf{a}_M(\boldsymbol{\tau}_r).$$



Condition Number in 3D

Theorem (3D bound)

With $\widetilde{\gamma}(\delta) = 2NP\delta^2 + 2NN_a\delta^2 + 2N\delta + PN_a\delta^2 + P\delta + N_a\delta + 1$,

$$\boxed{\kappa^2(\mathbf{W}) \leq \frac{MPN_a + \dfrac{\widetilde{\gamma}(\delta)}{\delta^3} - PN_a}{MPN_a - \dfrac{\widetilde{\gamma}(\delta)}{\delta^3} - PN_a}}$$

How we got there

Replace the 2D box $[-\delta, \delta]^2$ by the 3D box $[-\delta, \delta]^3$ in the Dirac-comb argument, use the $D \geq 2$ Beurling–Selberg bounds (Section II) and propagate the $1/\delta^3$ volume scaling.

Natural scaling

Same structure in 2D ($1/\delta^2$), 3D ($1/\delta^3$), and higher – the theory is *dimension-aware*.

Numerical Experiments

Experiment 1 – Condition Number Validation

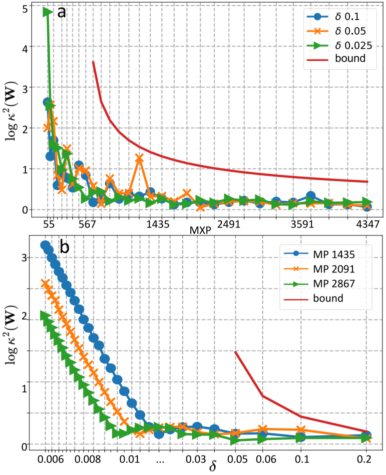
Setup

$\mathbf{B} = \mathbf{D}_i$ are random columns of a DFT matrix (*worst case*: highly correlated columns of $\mathbf{W}$). $J = 3$, $L = Q = 2$, $\mathbf{u}, \mathbf{v} \sim \mathcal{U}(0, 1]$.

- **(a)** Vary MP for $\delta \in \{0.025, 0.05, 0.1\}$.
- **(b)** Fix $MP = 2867$; sweep δ.

Take-away

Empirical $\kappa(\mathbf{W})$ lies *under* our theoretical bound for every (M, P, δ) tested; larger δ hits the well-conditioned regime with far fewer measurements.



Experiment 2 – Phase Transition vs. SoMAN

Protocol

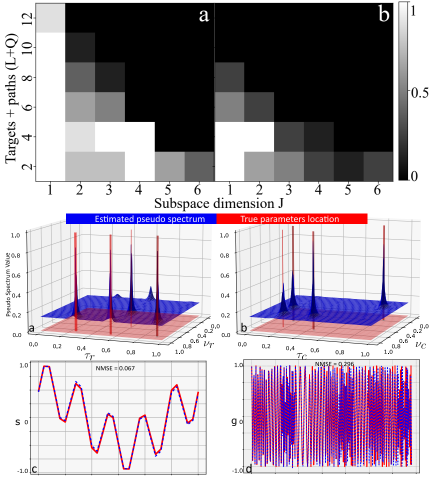
$P = 11$, $M = 31$, 20 Monte-Carlo trials. Success: $\|\widetilde{\mathbf{X}} - \mathbf{X}\|_F / \|\mathbf{X}\|_F < 10^{-1}$. Sweep subspace dimension J and $L + Q$ on both axes.

Result

The proposed low-rank CDH recovery achieves a visibly *higher* phase transition curve than SoMAN-SDP [?], for the same computational budget.

Single-instance example (noiseless)

$L = Q = 4$, $M = 31$, $P = 11$, $J = 2$. Parameters recovered to $\sim 10^{-3}$; waveform/message NMSE ≈ 0.30 and 0.30 respectively.



Experiment 3 – Noise Robustness

NMSE vs. M

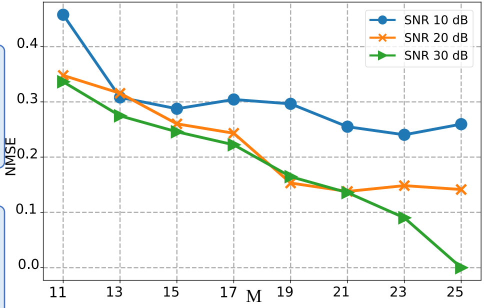
$P = 11$, $\delta = 0.1$, $J = 2$, $L = Q = 2$. SNR $\in \{10, 20, 30\}$ dB. NMSE of $\mathbf{X}$ decreases monotonically and saturates near the noise floor.

Parameter recovery under noise

For $\delta = 0.0125$:

- SNR 30 dB: $|\widehat{\tau}_r - \tau_r| \leq 10^{-3}$.
- SNR 10 dB: $\leq 10^{-2}$ on a grid of step 10^{-3}.

Message NMSE: 0.030 (30 dB), 0.034 (10 dB).



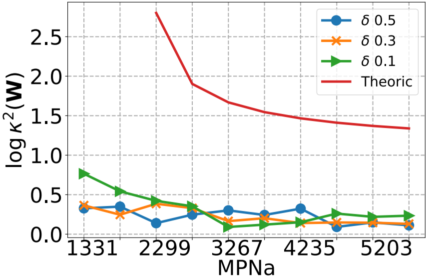
Experiment 4 – 3D Simulation

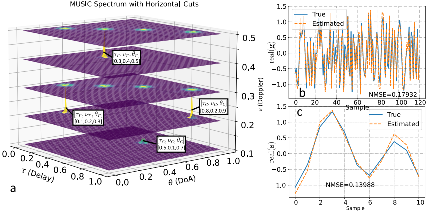
Setup

ULA with $N_a = 11,\ M = 11,\ P = 11,\ J = 2,$ $L = Q = 2.$ $\mathbf{B}$ random DFT columns (worst-case).

Observations

- $\kappa^2(\mathbf{W})$ below the 3D bound for $\delta \in \{0.1, 0.3, 0.5\}$ as M grows from 11 to 47.
- Horizontal cuts of the 4D pseudo-spectrum exhibit *exact* delay–Doppler–DoA localisation.
- NMSE$(\mathbf{s}) \approx 0.14$, NMSE$(\mathbf{g}) \approx 0.18$.





Summary

What we established

1. A **joint** delay–Doppler separation condition via Beurling–Selberg interpolation theory.
2. An explicit **condition-number bound** on the Vandermonde–Khatri–Rao matrix $\mathbf{W}$.
3. A **unified nuclear-norm** recovery of a single CDH matrix – better phase transition and lower cost than SoMAN–SDP.
4. A **three-dimensional** extension (delay–Doppler–DoA) with the same structural bound.

Open directions

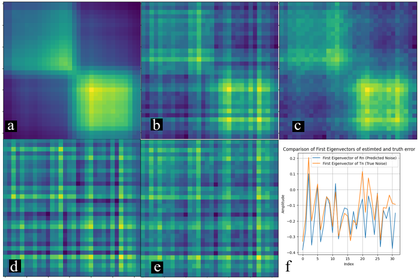
- Noise-perturbation bounds using second-order BS functions.
- Scaling to wideband / MIMO waveforms (cf. CLuP line of work).
- Complexity reduction of the SDP – reweighting, APG, block ℓ_1, low-rank Hankel factorisations.

Other Ongoing Works

Diffusion Models Gradient Preconditioning

Problem. in CS partitioning data into p subsets improves FIM conditioning but *injects noise* into the gradient.

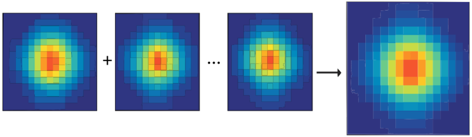
Key insight. The accumulation of gradient noise resembles a *forward diffusion process*: $\mathrm{Error}[\nabla \widetilde{f}(\boldsymbol{\Sigma})] = -\sum_{i=1}^{p} \mathbf{P}_i \mathbf{P}_i^T \mathbf{R}_i \mathbf{P}_i \mathbf{P}_i^T$. Train a CNN to *reverse* it.



Group Theory Radar Waveforms Design

Problem. Design *MIMO wideband* waveforms with low ambiguity-function sidelobes.

Key insight. The wideband AF is indexed by the *affine group* $(a, b) \in \mathbb{R}_+ \times \mathbb{R}$: $(U_{\mathcal{A}}(a, b)f)(t) = \frac{1}{\sqrt{a}} f\left(\frac{t-b}{a}\right)$. **Almost Difference Set** (ADS) $\rightarrow$ near-Welch-bound cross-correlation.



Thank you!

Questions & discussion

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