Binary Models for Opinion Dynamics and Interpersonal Relationships via Influence and Homophily Mechanisms

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Outline

- Motivation
- Part 1: A binary homophily based model
- Part 2: Intertwined homophily and influence based model
- Conclusion

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Time varying network dynamics

Nowadays opinions and interpersonal relationship evolve over time scales that are comparable in magnitude, especially in online social networks.

$\rightarrow$ both opinions and networks are **time-variant**

Understanding the network evolution leads to more reliable models that mimic the reaching of desired configurations that **minimize cognitive dissonance.**

Why Binary Models?

Only the **type** of relationships between individuals can be determined

$\rightarrow$ easier to infer

$\rightarrow$ people might not want to reveal intensity of relationships

$\rightarrow$ intensity of a relationships might be irrelevant

Part I
A Binary Homophily Model for Opinion Dynamics

Part I

- Model Description
- Equilibrium Points
- (V, Σ)-Factorization
- Non Structurally Balanced Equilibria

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Model Description

Let us consider a network made up of $N \geq 3$ agents.

$$x_{ij}(t) = \begin{cases} 1 & \text{if } i \text{ has a good opinion of } j \\ -1 & \text{if } i \text{ has a bad opinion of } j \end{cases}$$

$X(t) = [x_{ij}(t)]$, matrix in which the relationships among agents are collected.

Model Description

Upon defining

$$\mathcal{A}_{ij}(t) := \{k \in [1, N], k \neq i, j : x_{ik}(t)x_{jk}(t) = 1\},$$

$$\mathcal{D}_{ij}(t) := \{k \in [1, N], k \neq i, j : x_{ik}(t)x_{jk}(t) = -1\},$$

we have that

$$x_{ij}(t+1) = \begin{cases} 1, & \text{if } |\mathcal{A}_{ij}(t)| > |\mathcal{D}_{ij}(t)|; \\ -1, & \text{if } |\mathcal{A}_{ij}(t)| < |\mathcal{D}_{ij}(t)|; \\ x_{ij}(t), & \text{otherwise.} \end{cases}$$

Homophily mechanism: Individuals with similar opinions tend to be friendly to each other

Model Description

$$x_{ij}(t+1) = \begin{cases} 1, & \text{if } |\mathcal{A}_{ij}(t)| > |\mathcal{D}_{ij}(t)|; \\ -1, & \text{if } |\mathcal{A}_{ij}(t)| < |\mathcal{D}_{ij}(t)|; \\ x_{ij}(t), & \text{otherwise.} \end{cases}$$

In matrix form leads to

$$\mathsf{X}(t+1) = \mathrm{sign}\Big[\Big(\mathsf{X}(t) - \mathrm{diag}(\mathsf{X}(t))\Big)\Big(\mathsf{X}(t) - \mathrm{diag}(\mathsf{X}(t))\Big)^{\top} + \alpha \mathsf{X}(t)\Big]$$

Where $\mathsf{X}(t) = [x_{ij}(t)]_{i,j \in 1,\dots,N}$, $\alpha \in (0,1)$.

$$S_1^N := \{\mathsf{M} = \mathsf{M}^{\top} \in \{-1,1\}^{N \times N} : [\mathsf{M}]_{ii} = 1, \forall i \in [1,N]\},$$

Positively invariant set.

- Model Description
- Equilibrium Points
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Equilibrium Points

A matrix $\mathsf{X}^* \in S_1^N$ is an *equilibrium point* for the binary homophily model if

$$\mathsf{X}(0) = \mathsf{X}^* \Rightarrow \mathsf{X}(t) = \mathsf{X}^*, \ \forall t \geq 0.$$

Proposition

A matrix $\mathsf{X}^* \in S_1^N$ *is an equilibrium point if and only if* $\mathsf{X}^* = \mathrm{sign}((\mathsf{X}^*)^2 - \mathsf{X}^*)$.

Remark: If $\mathsf{X}^* \in S_1^N$ is such that $\mathcal{G}(\mathsf{X}^*)$ is structurally balanced, i.e. there exists $\mathsf{x} \in \{-1, 1\}^N$ such that $\mathsf{X}^* = \mathsf{x}\mathsf{x}^\top$, then the condition expressed above is trivially satisfied.

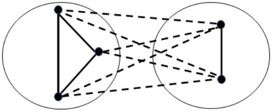

Figure: Complete structurally balanced network.

For certain values of N, the model admits only structurally balanced equilibrium points.

Proposition

Given a network of N agents, with $N \in \{3, 4, 5, 7\}$, if X^* is an equilibrium point for the binary homophily model, then $\mathcal{G}(\mathsf{X}^*)$ is structurally balanced.

Part I

An equivalent representation for the matrices in S_1^N that allows to derive additional conditions for the analysis of the equilibrium points.

Lemma

Given $\mathrm{X} \in S_1^N$, *there exist a permutation matrix* $\mathrm{P} \in \{0,1\}^{N \times N}$, *vectors* $\mathrm{v}_i \in \{-1,1\}^{n_i}, i \in [1,k]$, *and a matrix* $\Sigma \in S_1^k$, *such that*

$$\mathrm{P}^\top \mathrm{X} \mathrm{P} = \left[\begin{array}{c|c|c} \mathrm{v}_1 & & \\ \hline & \vdots & \\ \hline & & \mathrm{v}_k \end{array}\right] \underbrace{[\Sigma]}_{k \times k} \left[\begin{array}{c|c|c} \mathrm{v}_1^\top & & \\ \hline & \cdots & \\ \hline \underbrace{\phantom{\mathrm{v}_1^\top}}_{n_1} & & \underbrace{\mathrm{v}_k^\top}_{n_k} \end{array}\right].$$

Graph Interpretation

Consider $X^* = V\Sigma V^\top$

$$X^* = \begin{bmatrix} v_1 v_1^\top & \sigma_{12} v_1 v_2^\top & \dots & \sigma_{1k} v_1 v_k^\top \\ \sigma_{12} v_2 v_1^\top & v_2 v_2^\top & \dots & \sigma_{2k} v_2 v_k^\top \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1k} v_k v_1^\top & \sigma_{2k} v_k v_2^\top & \dots & v_k v_k^\top \end{bmatrix}$$

- The product $v_i v_i^\top, i \in [1, k]$, corresponds to a structurally balanced subclass, $\mathcal{C}_i$, with $n_i = |\mathcal{C}_i| = \dim(v_i)$, in $\mathcal{G}(X^*)$.
- The class $\mathcal{C}_i$ splits into two adverse factions, $\mathcal{C}_{iA}$ and $\mathcal{C}_{iB}$, each of them consisting of agents that are friends.
- σ_{ij} represents the relation between agents in class $\mathcal{C}_i$ and in $\mathcal{C}_j$.

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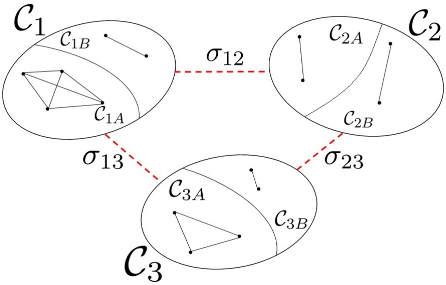

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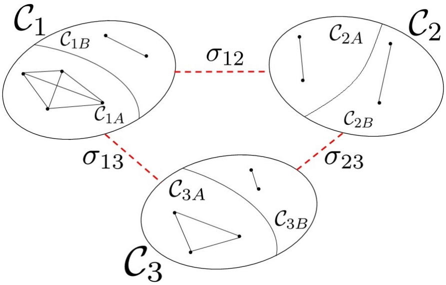

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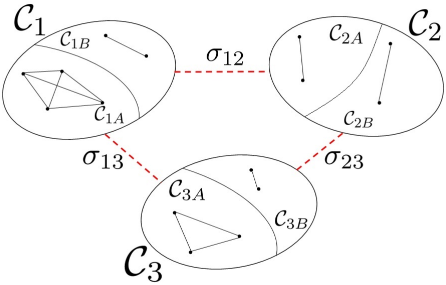

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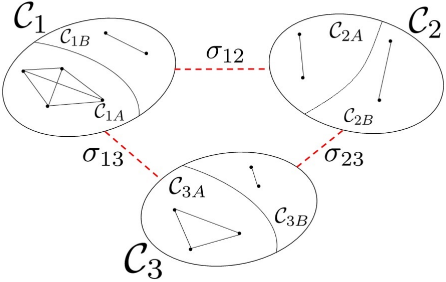

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Part I

Non Structurally Balanced Equilibria

Proposition

A matrix $\mathsf{X}^* \in S_1^N$ *is an equilibrium point for the binary homophily model if and only if*

$$\Sigma = \mathrm{sign}(\Sigma \mathbb{N} \Sigma - \Sigma),$$

where $\Sigma \in S_1^k$ *is the matrix* Σ *involved in a* (V, Σ)*-factorization of* X^**,* $\mathbb{N} := V^\top V = n_1 \oplus n_2 \oplus \cdots \oplus n_k$ *and* $n_1, \ldots, n_k$ *are the sizes of the vectors* v_i *appearing in* V*.*

Remark: For $N \geq 6$ there exist equilibrium points $\mathsf{X}^* \in S_1^N$ associated to a non structurally balanced graph $\mathcal{G}(\mathsf{X}^*)$.

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Non Structurally Balanced Equilibria

Proposition

If $\Sigma \in S_1^3$, *then one can always find positive integers* n_1, n_2, n_3 *such that*

$$\Sigma = \text{sign}(\Sigma \mathbb{N} \Sigma - \Sigma),$$

where $\mathbb{N} := n_1 \oplus n_2 \oplus n_3$.

If $\Sigma \in S_1^3$ fulfils $\Sigma = \text{sign}(\Sigma \mathbb{N} \Sigma - \Sigma)$ it is always possible to find a block-diagonal matrix V such that $X^* = V \Sigma V^\top$ is an equilibrium point.

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Recap

- A binary homphily model for the opinion dynamics has been proposed.
- Necessary and sufficient conditions for a configuration to be an equilibrium point have been provided.
  - All structurally balanced configurations are equilibrium points.
  - There can be structurally unbalanced equilibrium points.
- The (V, Σ)-factorization provides an alternative characterization of all the equilibrium points.
- Social networks of sufficiently large size N exhibit equilibrium points whose graph splits into three structurally balanced classes.

Part II
A Bandwagon Bias Based Model for Opinion Dynamics: Intertwining between Homophily and Influence Mechanisms

Part II

- The model
- Equilibrium points
- Convergence properties
- Simulations
- Conclusion

Part II

Objective

Interpersonal appraisals and individual opinions evolve in an intertwined way: mathematical abstraction of **bandwagon bias**.

The model constitutes the mixed real/binary version of a model presented in [1] [1].

[1] [1] F. Liu, S. Cui, W. Mei, F. Dörfler, and M. Buss. Interplay between homophily-based appraisal dynamics and influence-based opinion dynamics: Modeling and analysis. IEEE Control Systems Letters, 5(1):181–186, 2020.

The bandwagon bias

Bandwagon bias is a cognitive bias according to which our opinions on topics are influenced by the opinions that other individuals have on the same topics and by the relationships we have with those individuals.

The model

Given a group of N agents

- $X(t) = X^\top(t) \in \{-1, 0, 1\}^{N \times N}$, appraisal network at time t
  - $X_{ij}(t) = 1/0/-1$ if i has a positive/neutral/negative feeling towards j at time t

- $Y(t) \in \mathbb{R}^{N \times m}$, opinion matrix at time t
  - $Y_{ij}(t)$ opinion agent i has on topic j, at time t.

The model

Opinion matrix and appraisal matrix evolve in an intertwined way

$$X_{ij}(t+1) = \operatorname{sgn}\left(\sum_{k=1}^{m} Y_{ik}(t) Y_{jk}(t)\right)$$
$$Y_{ij}(t+1) = \frac{1}{N}\sum_{k=1}^{N} X_{ik}(t+1) Y_{kj}(t).$$

that in matrix form reads as

$$\mathsf{X}(t+1) = \operatorname{sgn}(\mathsf{Y}(t)\mathsf{Y}(t)^\top)$$
$$\mathsf{Y}(t+1) = \frac{1}{N}\mathsf{X}(t+1)\mathsf{Y}(t)$$

The two mechanisms

$$X_{ij}(t+1) = \operatorname{sgn}\left(\sum_{k=1}^{m} Y_{ik}(t)Y_{jk}(t)\right)$$

$$Y_{ij}(t+1) = \frac{1}{N}\sum_{k=1}^{N} X_{ik}(t+1)Y_{kj}(t).$$

- **Homophily mechanism:** Individuals with similar opinions tend to be friendly to each other
- **Influence mechanism:** Each individual's opinion is influenced by the others' opinions proportionally to her/his appraisal of them.

The model

The two equations can be grouped in

$$\mathsf{Y}(t+1) = \frac{1}{N}\mathrm{sgn}(\mathsf{Y}(t)\mathsf{Y}(t)^\top)\mathsf{Y}(t).$$

The mathematical abstraction of the bandwagon bias leads to a peculiar form of opinion dynamics model.

Assumption: $Y(0)$ is devoid of zero columns/rows.

Part II

Equilibrium points

Definition

A pair (Y^*, X^*) is an equilibrium point for the model if

$$
\begin{aligned}
X^* &= \operatorname{sgn}(Y^*(Y^*)^\top) \\
Y^* &= \frac{1}{N} X^* Y^*.
\end{aligned}
$$

Proposition

A pair $(\mathrm{Y}^*, \mathrm{X}^*) \neq (0,0)$ *is an equilibrium point for the model if and only if*

i) $\mathrm{X}^* = \mathrm{p}\mathrm{p}^\top$, *for some* $\mathrm{p} \in \{-1, 1\}^N$;

ii) $\mathrm{Y}^* = \mathrm{p}\begin{bmatrix} a_1 & a_2 & \dots & a_m \end{bmatrix}$, *for some* $a_i \in \mathbb{R}$, $\sum_{i=1}^m a_i^2 \neq 0$.

$X^* = pp^\top$ corresponds to a complete structurally balanced network.

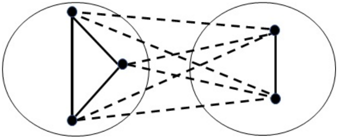


Figure: Complete structurally balanced network.

The sign distribution of opinions mirrors the network partition into factions (polarization!).

Part II

Convergence properties

$$Y(t+1) = \frac{1}{N}\mathrm{sgn}(Y(t)Y(t)^\top)Y(t)$$

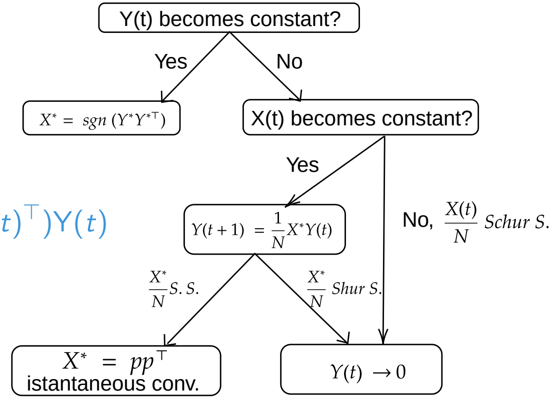


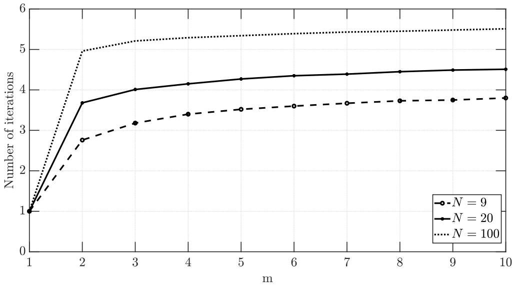
Figure: Flow-chart for convergence.

Part II

Simulations

Monte Carlo simulations validate the convergence properties of the model.

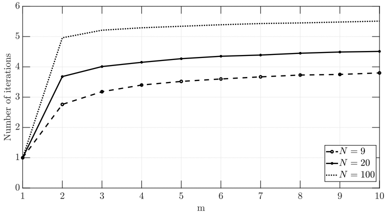
The figure shows how the average number of iterations needed to reach a structural balanced configuration over the total number of 30000 simulations varies as a function of the number of topics $m \in [1, 10]$, for networks involving $N = 9, 20, 100$ agents, $Y_{ij}(0) \sim \mathcal{N}(0, 100)$.



Simulations

$$P(|\hat{p} - p| \leq \epsilon) \geq 1 - \delta$$

- accuracy $\epsilon = 0.01$
- confidence level $1 - \delta = 0.99$
- Estimated probability $\hat{p}$ to reach a structurally balanced configuration,
  - $\hat{p} = 1$, for $N = 20, 100$, $\forall m \in [1, 10]$
  - $\hat{p} \geq 0.98$ $N = 9$, $\forall m \in [1, 10]$



Part II

Recap

We have proposed a modified version of the model in [1][2] for the interplay between homophily-based appraisal dynamics and influence-based opinion dynamics.

Our model is easier to validate on real data, since we only rely on the type of interactions.

Nontrivial equlibria can always be reached in a finite number of steps. The case when all opinions and appraisals converge to zero corresponds to sets of initial conditions of zero measure.

[2] [1] F. Liu, S. Cui, W. Mei, F. Dörfler, and M. Buss. Interplay between homophily-based appraisal dynamics and influence-based opinion dynamics: Modeling and analysis. IEEE Control Systems Letters, 5(1):181–186, 2020.

Thanks for your attention!