Spreading Processes over Networks

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Talk Outline

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Examples of spreading processes

(a) Diseases

(b) Opinions

(c) Computer viruses

Consequences of disease spread

(a) Black death 1347-1352

(b) Small pox 1633

Be a Booster for the Fourth Liberty Loan Bond Sale; Help Your Country Win the War

BEMIDJI DAILY PIONEER

PUBLIC PLACES ARE ORDERED CLOSED

GERMANY'S REPLY HANDED TO SWISS MINISTER; KAISER'S ALLIES WOULD QUIT

MAYOR ISSUES ORDER TO PREVENT SPREAD INFLUENZA PLAGUE

Schools, Churches, Motion Picture Theatres and All Places of Public Gathering Come Under Ban, Closing Period Indefinite

(c) Spanish Flu, 1918 — 1920

The Straits Times

SCHOOLS MAY HAVE TO CLOSE—25,000 PUPILS ABSENT

Health and Education Ministries making urgent review of the situation

SINGAPORE FLU SCARE

Worst ever in Colony history

Chick-in-egg patient ('not cured') is out now

The 'king' and I—by Onn

'HOLD TIGHT TO YOUR REINS'

PARTY WALKS OUT FROM CONGRESS

SINGAPORE 'THE FRONT LINE'

You can't go wrong with Sissons

(d) Asian Flu 1957 — 1968

Questions of interest

- How can we model the spread of a disease in a community?
- How to leverage the tools from systems theory to address an epidemic outbreak?

Talk Outline

1. Spreading Processes
2. Models
3. SIS
4. SIS - Competition
5. SIS-Co-operative case

Popular Models

- Divide the population into various compartments $\rightarrow$ Kermack & McKendrick'32

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- S - Susceptible, I - Infected, R - Recovered, E - Exposed, A - Asymptomatic

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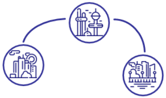
- S - Susceptible, I - Infected, R - Recovered, E - Exposed, A - Asymptomatic
- SIR →Van Mieghem'14, W. Mei et al'17
- SEIR → Li & Muldowney'95, Arcede et al'20
- SAIR → Hethcote'00, Mena-Lorca & Hethcote'92
- SIS → Lajmanovich & Yorke'76, Fall et al'07, Khanafer et al'16
  This talk!

Talk Outline

Modeling of a Virus Spread over Networks

Primer on SIS Model

- Population - Collection of individuals staying in the same district/city.
- A disease may spread from one population to another.



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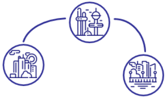
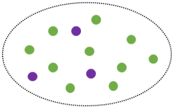
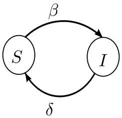


▷ Susceptible-Infected-Susceptible (SIS) Model

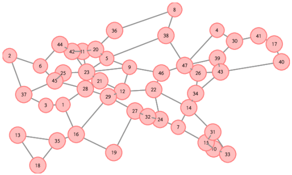
- A single well-mixed population of individuals with fixed size
- An individual is either Susceptible (S) or Infected (I)
- A susceptible individual, depending on its infection rate β, gets infected; an infected individual, depending on its healing rate δ, recovers
- Extensive literature - When disease gets eradicated? How to eradicate the disease?





▷ Networked SIS model $\rightarrow$ multiple population nodes; interconnection described by some graph

Primer on SIS model- contd.



N/w of cities in the EU

$x_i(t)$: fraction of population in node i infected at time t

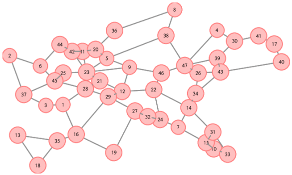
$$\dot{x}_i(t) = (1 - x_i(t))\beta_i \sum_{j=1}^{n} a_{ij}x_j(t) - \delta_i x_i(t))$$

$$\dot{x}(t) = (-D + (I - X)\bar{B}A)x(t)$$

$x = 0 \implies$ population is healthy

$\mathbf{x}^\star$ - endemic

Primer on SIS model- contd.



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Advantages

When the epidemic gets eradicated (stability!)?
How to eradicate the epidemic (control!)?

State of the art and contributions

Time-invariant networks

Lajmanovich & Yorke'76; Mieghem et al'09; Khanafer et al'16

Time-varying networks

Rami et al'14; Paré et al'18;
Gracy et al, IEEE-TCNS'20; Paré, Gracy et al CDC'20; Gracy et al, ECC'22; Gracy et al, IFAC-WC 2023

Key Terminologies

1. Define $B := \bar{B}A$
2. Define $\mathcal{R}_0 := \rho(D^{-1}B)$ - this is known as the the reproduction number
3. If $\mathcal{R}_0 \leq 1$, then the disease gets eradicated
4. If $\mathcal{R}_0 > 1$, then the disease becomes endemic

Talk Outline

Multi-competitive viruses

1. What are competitive viruses?

Multi-competitive viruses

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2. Competing spreading processes occur in many contexts:
   1. Spread of competing viral strains

Rhinovirus vs influenza A
→ A. Wu at al The Lancet Microbe'20
spread of influenza A virus during 2009 might have been interrupted by the annual autumn rhinovirus epidemic.

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How do we study such processes?

Spread of multi-competitive viruses

Historical Context (two virus case)

1. Different strains of gonorrhea emerged in the (late) 70's → Hethcote & Yorke'84
   From 200, 000 cases/year in the 50's to 1, 000, 000 cases/year in the late 70's

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2. Two population case $\longrightarrow$ Novak'91, Castillo-Chavez et al'95, '96
3. Three population case (but only for the tree structure) $\longrightarrow$ Castillo-Chavez et al'99
4. Multi-population case
   - the undirected graph case $\longrightarrow$ Sahneh & Scoglio'14
   - regular graphs $\longrightarrow$ Santos et al'15
   - directed graph $\longrightarrow$ J. Liu et al'19, A. Janson et al'20, M. Ye et al'22, Anderson & Ye'23
   - directed and time-varying graph $\longrightarrow$ Paré et al'20

Spread of multi-competitive viruses

Modeling

Multi-competitive Networked SIS Model

- Let n denote the number of population nodes.
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No individual can be infected with more than one virus at the same time

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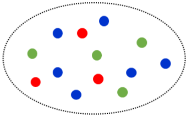
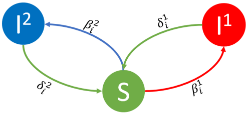
- A population node is healthy if all individuals belong to the S compartment; otherwise, it is infected
- An individual in the susceptible compartment, can transition to the "infected with virus k" compartment at a rate β_i^k
- An individual in population i that is infected with virus k recovers from it based on its healing rate with respect to virus k, i.e., δ_i^k



A possible scenario within a population node



Visualization of the multi-competitive SIS model for the case when there are two competing viruses.

Multi-competitive viruses

- Spread over a network of populations can be captured using a graph
- Let G be a 2-layer graph

  - vertices- the population nodes.
  - E^k- edge set for the k^{th} layer
  - A^k - weighted adjacency matrix for the k^{th} layer
  - $(i, j) \in E^k$ if, and only if, $a^k_{ji} \neq 0$

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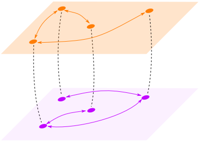

Visualization of graph G for 2-layer case. In yellow, layer corresponding to spread of virus 1; in magenta, layer corresponding to spread of virus 2

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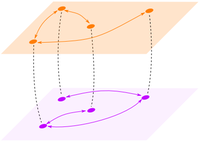

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Challenges

- Multi-virus spread exhibit far richer dynamics

$$\dot{x}^1(t) = \Big(\big(I - X^1(t) - X^2(t)\big) B^1 - D^1 \Big) x^1(t)$$
$$\dot{x}^2(t) = \Big(\big(I - X^1(t) - X^2(t)\big) B^2 - D^2 \Big) x^2(t)$$

Competitive bivirus setting

What is the limiting behavior of these systems?

Question

Given an initial state, do the dynamics converge (or not) to some equilibrium?

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Answer [M.Ye et al SICON 2022]

- For almost all initial states, yes! In fact, if the Jacobian is irreducible Metzler, these converge to a stable equilibrium!
- Set of initial states that do not converge to some stable equilibrium has measure zero.

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Why is this so?

1. Bi-virus system is monotone
2. Furthermore, for almost all choices of healing and infection rates, the system has a finite number of equilibria
3. $1 + 2 \implies$ Answer; see [Hal Smith, 1988]

Multi-competitive viruses

Bi-virus systems and monotone dynamical systems

- It turns out the bi-virus system is monotone
- Consider system ⋆

$$\dot{x}(t) = f(x(t))$$

- Notion of monotonicity[1]
  - Let $\phi(x_0, t)$ denote the solution of $x(t)$ for the system ⋆, with initial conditions x_0
  - Assume $f(x)$ is "sufficiently smooth"

Then, $x_0 \leq y_0$ implies $\phi(x_0, t) \leq \phi(y_0, t)$ for all $t \geq 0$

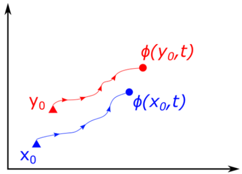

Fig. courtesy Dr. Mengbin Ye

[1] Morris W. Hirsch (SIAM'82, '85) established powerful convergence results for a class of nonlinear systems known as monotone systems, and many subsequent results have followed; for a classical review, see Hal Smith SIAM'88.

Healthy state analysis

- The state $(\mathbf{0}, \mathbf{0})$ is referred to as the healthy state.
- Define $\mathcal{R}_0^1 = \rho((D^1)^{-1}B^1)$, and $\mathcal{R}_0^2 = \rho((D^2)^{-1}B^2)$

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Main Result

- If $\mathcal{R}_0^1 < 1$ and $\mathcal{R}_0^2 < 1$, then the dynamics converge to the healthy state exponentially fast; see [A. Janson, S.Gracy et al arxiv'20].
- If $\mathcal{R}_0^1 \leq 1$ and $\mathcal{R}_0^2 \leq 1$, then the healthy state is the unique equilibrium of the system. Furthermore, it is globally asymptotically stable; see [J. Liu et al TAC'19].

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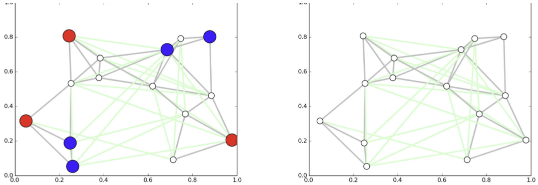

Fig. courtesy P. E. Paré.
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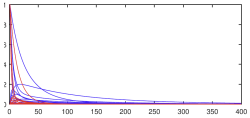

Fig. courtesy P. E. Paré.
Blue - infection levels corresponding to virus 2; red - infection levels corresponding to virus 1

Persistence of virus

One virus alive; other dead

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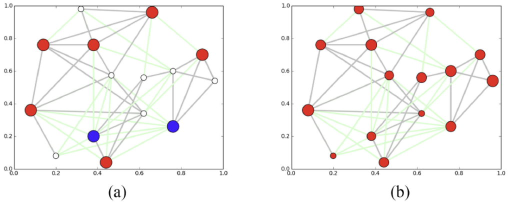

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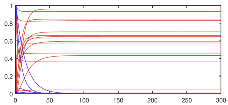

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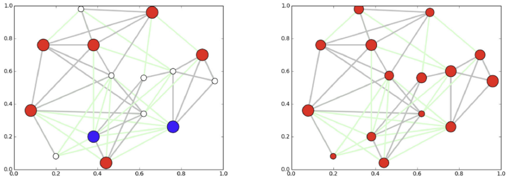

(a) (b)

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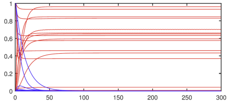

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Consequence

- The healthy state is the unique equilibrium $\iff$ $\mathcal{R}_0^1 \leq 1$ and $\mathcal{R}_0^2 \leq 1$

Persistence of virus

Both viruses alive - coexistence case for nongeneric networks

What happens if both of the aforesaid eigenvalue conditions were to be violated? - Several possibilities!

- If $\mathcal{R}_0^1 > 1$ and $\mathcal{R}_0^2 > 1$, then there exists exactly two single-virus endemic equilibria $(\tilde{x}^1, \mathbf{0})$ and $(\mathbf{0}, \tilde{x}^2)$, such that $\mathbf{0} \ll \tilde{x}^1 \ll \mathbf{1}$ and $\mathbf{0} \ll \tilde{x}^2 \ll \mathbf{1}$.

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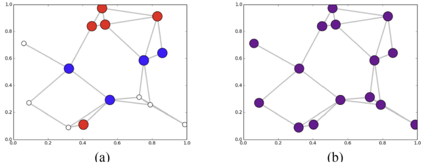
Line of coexistence equilibria - homogenous spread

Let A denote the weighted adjacency matrix of the graph. Suppose that the viruses spread over the same network, $\delta_i^1 = \delta^1$, $\delta_i^2 = \delta^2$, $\beta_{ij}^1 = \beta^1$ and $\beta_{ij}^2 = \beta_2$. Suppose further that $s(A) > \frac{\delta^1}{\beta^1} = \frac{\delta^2}{\beta^2}$. If $(\tilde{x}^1, \tilde{x}^2)$ with $\tilde{x}^1, \tilde{x}^2 > \mathbf{0}$ is an equilibrium, then $\tilde{x}^1, \tilde{x}^2 \gg \mathbf{0}$, and $\tilde{x}^1 = \alpha\tilde{x}^2$ for some $\alpha > 0$.



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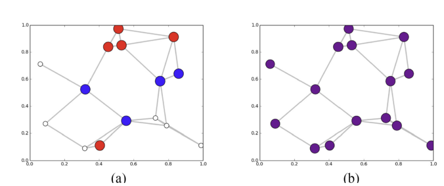

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Line of coexistence equilibria-extension

The result can be extended for heterogeneous spread, but the viruses have to be identical, and spread over the same graph.

Persistence of virus

Both viruses alive - coexistence case for generic networks, part a)

Could we do with less restrictions on the model parameters?

Persistence of virus

Both viruses alive - coexistence case for generic networks, part a)

Could we do with less restrictions on the model parameters? Yes!

Existence of a coexistence equilibrium point $\rightarrow$ A. Janson, S.Gracy et al, IEEE-TNSE'23, Under Review

Suppose that $\mathcal{R}_0^1 > 1$ and $\mathcal{R}_0^2 > 1$. If

- $s(-D^1 + (I - \text{diag}(\tilde{x}^2))B^1) > 0$
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then there exists at least one coexisting equilibrium $(\hat{x}^1, \hat{x}^2) \gg \mathbf{0}$ such that $\hat{x}^1 + \hat{x}^2 \ll \mathbf{1}$.

Is the coexistence eqm. unique? NO - Doshi et al, IEEE/ACM ToN'22.

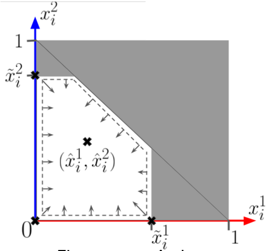

Fig. courtesy A. Janson

Persistence of virus

Both viruses alive - coexistence case for generic networks, part a)

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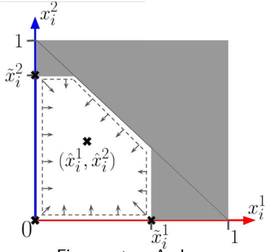

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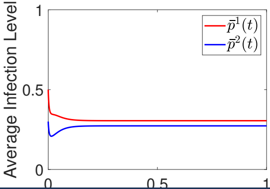

In simple English....

Both viruses exist in some sort of balance with each other!

Persistence of virus

Both viruses alive - coexistence case for generic networks, part-b)

What about other stability configurations for the boundary equilibria?

Persistence of virus

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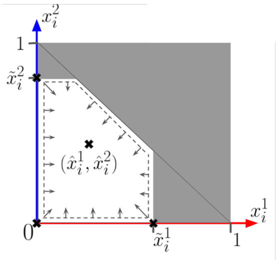

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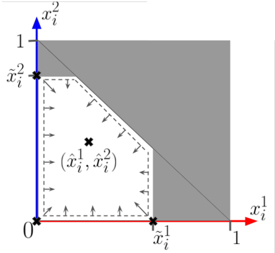

What happens if one is stable and the other is unstable?

No co-existence equilibrium exists...

Persistence of virus

Both viruses alive - coexistence case for generic networks, part-b)

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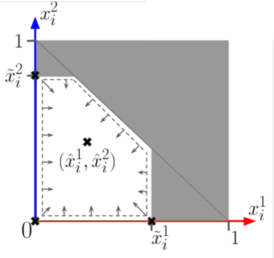

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How many coexistence equilibria exist?

Lower bound on the number of co-existence equilibria → BDO Anderson, M. Ye, SIAM ADS 2023.

Persistence of virus

Both viruses alive - competitive exclusion case

Competitive exclusion

If two viruses are present such that one has a slight advantage over the other, then the virus with the advantage pushes out the other virus eventually.

Persistence of virus

Both viruses alive - competitive exclusion case

Competitive exclusion

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When does competitive exclusion occur?

- If $B^2 > B^1$,then there are exactly 3 equilibria, namely i) the healthy state $(\mathbf{0}, \mathbf{0})$, which is unstable; ii) the boundary equilibria $(\bar{x}^1, \mathbf{0})$, which is unstable, and iii) the boundary equilibria $(\mathbf{0}, \bar{x}^2)$, which is locally exponentially stable. $\rightarrow$ A. Janson, S. Gracy et al. arxiv'20; M. Ye at al, SICON'22

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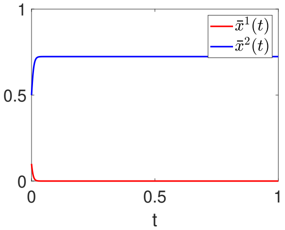

Blue virus pushes out red virus

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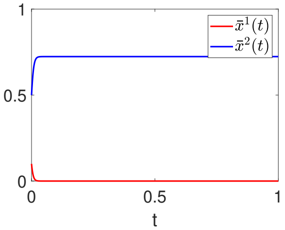

Blue virus pushes out red virus

Implications for control of epidemics

- Two strains of a virus.
- One benign; other malignant
- Drive the population to the single-virus endemic equilibrium rather than the healthy state

Competitive Exclusion

Who excludes who? Is the identity of the winner always the same?

- Recall that if $B^2 > B^1$, then virus 2 always beats virus 1
- Note that this is independent of the initial state

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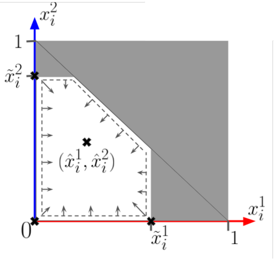

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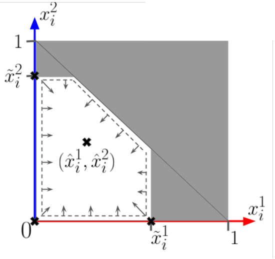


We want both $(\bar{x}^1, \mathbf{0})$ and $(\mathbf{0}, \bar{x}^2)$ to be locally exponentially stable at the same time.

Answer

- YES! We can! $\rightarrow$ M. Ye, BDO Anderson, A. Janson, S. Gracy, KH Johansson, under review at Systems and Control Letters'23.
- For almost any network layer of one virus, there exists a network layer for the other virus such that the resulting two-layer network satisfies a necessary and sufficient condition for each of the boundary equilibrium to be simultaneously stable
- $\implies$ Convergence to either boundary equilibria depends on the initial state.

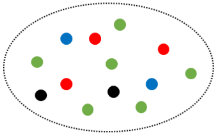
What happens if there are more than 2 competing viruses?

Multi-competitive viruses- how well do things generalize

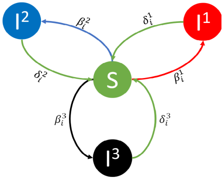
Modeling

Multi-competitive Networked SIS Model

- Let n denote the number of population nodes.
- Assuming there are 3 viruses spreading, an individual is either Susceptible (S) or Infected with virus 1 (I^1) or Infected with virus 2 (I^2) or Infected with virus 3 (I^3).



A possible scenario within a population node.



Visualization of the competitive tri-virus SIS model

Multi-competitive viruses- how well do things generalize? and other results

▷

$$\dot{x}^k(t) = \Big(- D^k + \big(I - \textstyle\sum_{l=1}^{3} \mathrm{diag}(x^l(t))\big) B^k \Big) x^k(t), \text{where } k = 1, 2, 3 \tag{1}$$

Multi-competitive viruses- how well do things generalize? and other results

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- The tri-virus system is NOT monotone $\longrightarrow$ Gracy et al, ACC, 2023
  Proof Sketch: Write the Jacobian of the system. Construct a signed graph based on the entries in the Jacobian. Show that there is a cycle with an odd number of negative signs.
- So what?
  - No idea what the limiting behavior is.
  - No dynamical behavior, including chaos, can be definitively ruled out.

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- Finiteness of equilibria is retained → Gracy et al, SIAM Journal of Applied Dynamical Systems, Under Review, 2023

- Various Equilibria:
  1. Disease-free equilibrium (DFE) - $(\mathbf{0}, \mathbf{0}, \mathbf{0})$.
  2. Boundary equilibria - $(\mathbf{0}, \ldots, \tilde{x}^k, \ldots, \mathbf{0})$, with $\mathbf{0} \ll \tilde{x}^k \ll \mathbf{1}$.
  3. Coexistence equilibria - $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$, where at least $\bar{x}^i$ and $\bar{x}^j$ $(i, j \in [3], i \neq j)$ are nonnegative vectors with at least one positive entry in each of $\bar{x}^i$ and $\bar{x}^j$.

Local exponential convergence to a boundary equilibrium

Theorem

[a] *Consider the tri-virus system. The boundary equilibrium* $(\tilde{x}^1, \mathbf{0}, \mathbf{0})$ *is locally exponentially stable if, and only if, each of the following conditions are satisfied:*

1. $\rho((I - \tilde{X}^1)(D^2)^{-1}B^2) < 1$; *and*
2. $\rho((I - \tilde{X}^1)(D^3)^{-1}B^3) < 1$.

If $\rho((I-\tilde{X}^1)(D^2)^{-1}B^2) > 1$ *or if* $\rho((I-\tilde{X}^1)(D^3)^{-1}B^3) > 1$, *then* $(\tilde{x}^1, \mathbf{0}, \mathbf{0})$ *is unstable.*

[a] S. Gracy, M. Ye, BDO Anderson, C Uribe, ACC'23.

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Proof Sketch:

$$J(\tilde{x}^1, \mathbf{0}, \mathbf{0}) = \begin{bmatrix} -D^1 + (I - \tilde{X}^1)B^1 - \hat{B}^1 & -\hat{B}^1 & -\hat{B}^1 \\ \mathbf{0} & -D^2 + (I - \tilde{X}^1)B^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -D^3 + (I - \tilde{X}^1)B^3 \end{bmatrix},$$

where $\hat{B}^i = \mathrm{diag}(B^i \tilde{x}^i)$, for $i = 1, 2, 3$

Existence and attractivity of a continuum of equilibria for nongeneric tri-virus networks

- Let z denote the single-virus endemic equilibrium corresponding to virus 1, with $Z = \text{diag}(z)$.
- Hence, assuming $D^1 = I$, the vector z fulfills the following:

$$-I + ((I - Z)B^1)z = \mathbf{0}. \tag{2}$$

- Let C be any nonnegative irreducible matrix for which z is also an eigenvector corresponding to eigenvalue one. That is, $Cz = z$.
- Define

$$B^2 := (I - Z)^{-1}C. \tag{3}$$

Existence and attractivity of a line of equilibria

Theorem

[a] *Consider the tri-virus system. Suppose that $D^k = I$ for $k \in [3]$. Suppose that B^1 and B^3 are arbitrary nonnegative irreducible matrices; and vector z and matrix B^2 are as defined in (4) and (5), respectively. Then, a set of equilibrium points of the tri-virus equations is given by $(\beta_1 z, (1-\beta_1)z, \mathbf{0})$ for all $\beta_1 \in [0,1]$. Furthermore,*

1. *if $s(-I + (I - Z)B^3) < 0$, then the equilibrium set $(\beta_1 z, (1-\beta_1)z, \mathbf{0})$, with $\beta_1 \in [0,1]$, is locally exponentially attractive.*
2. *if $s(-I + (I - Z)B^3) > 0$, then the equilibrium set $(\beta_1 z, (1-\beta_1)z, \mathbf{0})$, with $\beta_1 \in [0,1]$, is unstable.*

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[a] S. Gracy, M. Ye, BDO Anderson, C Uribe, ACC'23.

Proof Sketch:

1. Note that the Jacobian is block upper triangular.

$$J(\beta_1 z, (1-\beta_1)z, \mathbf{0}) = \begin{bmatrix} -I + (I-Z)B^1 - \mathrm{diag}(B^1\beta_1 z) & -\mathrm{diag}(B^1\beta_1 z) & -\mathrm{diag}(B^1\beta_1 z) \\ -\mathrm{diag}(B^2(1-\beta_1)z) & -I + (I-Z)B^2 - \mathrm{diag}(B^2(1-\beta_1)z) & -\mathrm{diag}(B^2(1-\beta_1)z) \\ \mathbf{0} & \mathbf{0} & -I + (I-Z)B^3 \end{bmatrix}$$

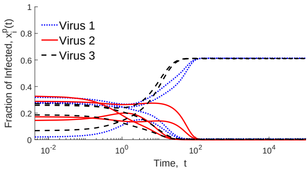
2. Find a coordinate transform such that the 11-block is irreducible Metzler.
3. Use Center-Manifold theorem.

Coexistence equilibria

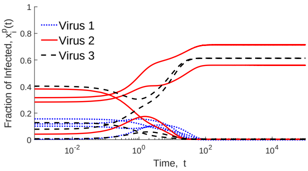
2-coexistence - two viruses alive, one dead

Proposition

Consider the tri-virus system. Suppose that $\rho((D^k)^{-1}B^k) > 1$ *for* $k \in [3]$*. There exists at least one 2-coexistence equilibrium, i.e.,* $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ *where* $\mathbf{0} \ll \bar{x}^i, \bar{x}^j \ll \mathbf{1}$ *for some* $i, j \in [3], i \neq j$*, and, for* $\ell \neq i, \ell \neq j$, $\bar{x}^\ell = \mathbf{0}$.



Virus 2 is dead



Virus 1 is dead

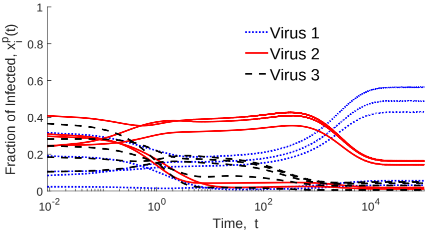
Coexistence equilibria

3-coexistence - all three viruses alive

Theorem

[a] *Consider tri-virus system. Suppose that, for each* $k \in [3]$, $\rho((D^k)^{-1}B^k) > 1$. *Suppose that every boundary equilibrium is unstable. Suppose that every 2-coexistence equilibrium is unstable. Then there exists at least one 3-coexistence equilibrium*

[a] S. Gracy, M. Ye, BDO Anderson, CA Uribe, SIAM Journal of Applied Dynamical Systems'23, Under Review



All three viruses are present in each population node.

Talk Outline

Co-operative Viruses

Motivation + model

Covid: Woman aged 90 died with double variant infection

By Michelle Roberts
Health editor, BBC News online

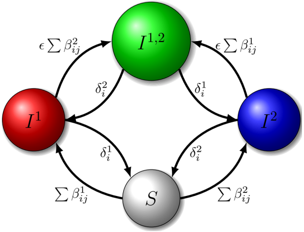
11 July 2021

Coronavirus pandemic

Infection with HIV increases the chances of contracting syphilis and vice-versa $\rightarrow$ Newman, M. E. and Ferrario, C. R. (2013). Interacting epidemics and coinfection on contact networks. PloS one

Co-operative Viruses

Model



Coupled bivirus SIS model

- $x_i^1(t)$: fraction of individuals in node i infected only with virus 1
- $x_i^2(t)$: fraction of individuals in node i infected only with virus 2
- $z_i(t)$: fraction of individuals in node i infected with both viruses 1 and 2

Co-operative Viruses

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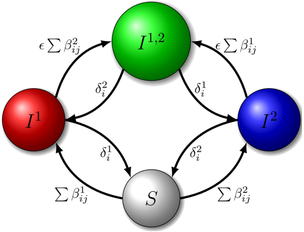

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Equations

$$\begin{aligned}
\dot{x}^1(t) &= -D^1x^1(t) + D^2z(t) + (I - X^1(t) - X^2(t) - Z(t))B^1(x^1(t) + z(t)) \\
&\quad -\epsilon^1X^1(t)B^2(x^2(t) + z(t)), \\
\dot{x}^2(t) &= -D^2x^2(t) + D^1z(t) + (I - X^1(t) - X^2(t) - Z(t))B^2(x^2(t) + z(t)) \\
&\quad -\epsilon^2X^2(t)B^1(x^1(t) + z(t)), \\
\dot{z}(t) &= -(D^1 + D^2)z(t) + \epsilon^1X^1(t)B^2(x^2(t) + z(t)) + \epsilon^2X^2(t)B^1(x^1(t) + z(t)).
\end{aligned}$$

Co-operative viruses

- ϵ^k- the *coupling parameters* between the two viruses.

  Various paradigms
  - If $\epsilon^k \in (0, 1)$, that means if a node is infected with virus k it is less susceptible to the other virus $\rightarrow$ Suppressing regime.
  - If $\epsilon^k > 1$ that means if a node is infected with virus k, it is more likely to become infected with the other virus $\rightarrow$ Reinforcing regime.
- How does it connect to other models that we have seen?
  - If $\epsilon^k = 0$ for $k = 1, 2$, then the co-operative virus model coincides with the competitive bi-virus model.
  - If $\epsilon^k = 1$ for $k = 1, 2$, then the co-operative virus model is the same as two independent single-virus SIS systems.

Co-operative viruses

- The point $(\mathbf{0}, \mathbf{0}, \mathbf{0})$ is an equilibrium of the system. We call it the healthy state.
- Any non-zero equilibrium is referred to as an endemic equilibrium.

Main Result S. Gracy et al'22, Automatica, Under Revision

- The healthy state is locally exponentially stable if, and only if, $s(-D^1 + B^1) < 0$, $s(-D^2 + B^2) < 0$, and $\delta_i^1 + \delta_i^2 > 0$ for all $i \in [n]$. If $s(-D^1 + B^1) > 0$ or if $s(-D^2 + B^2) > 0$, then the healthy state is unstable, and there exists an endemic equilibrium.

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- If $\epsilon^1, \epsilon^2 \in [0, 1]$, $s(B^1 - D^1) \leq 0$ and $s(B^2 - D^2) \leq 0$, then the healthy state is the unique equilibrium of the system, and the system asymptotically converges to the healthy state for any initial state.

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Take home messages

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- SIS models allow for the possibility of re-infection.
- Multi-virus dynamics are more complicated than single-virus dynamics:
  - Possibility of existence of co-existence eqm.
    - both the boundary equilibria being unstable
    - both the boundary equilibria being stable
  - Competitive exclusion
  - When two viruses compete, the system is monotone; when three (or more) viruses compete, the system is not monotone.

Collaborators

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Some ongoing work + Problems of possible interest

1. How to factor in interactions beyond pairwise interactions [2]
2. How to account for scenarios where the total population size is not constant
3. Design (decentralized) feedback controllers to eradicate a disease
4. Given some disease-related data, can we learn the network topology and disease dynamics
5. Resource allocation (at the nodes and/or at the edges) for controlling bivirus spread

[2] S. Gracy et al, Competitive networked bivirus SIS spread with higher order interactions, Submitted to ACC'24

Existence and attractivity of a continuum of equilibria for nongeneric tri-virus networks

- Let z denote the single-virus endemic equilibrium corresponding to virus 1, with $Z = \text{diag}(z)$.
- Hence, assuming $D^1 = I$, the vector z fulfills the following:

$$-I + ((I - Z)B^1)z = \mathbf{0}. \tag{4}$$

- Let C be any nonnegative irreducible matrix for which z is also an eigenvector corresponding to eigenvalue one. That is, $Cz = z$.
- Define

$$B^2 := (I - Z)^{-1}C. \tag{5}$$

Existence and attractivity of a line of equilibria

Theorem

[a] *Consider the tri-virus system. Suppose that $D^k = I$ for $k \in [3]$. Suppose that B^1 and B^3 are arbitrary nonnegative irreducible matrices; and vector z and matrix B^2 are as defined in (4) and (5), respectively. Then, a set of equilibrium points of the tri-virus equations is given by $(\beta_1 z, (1-\beta_1)z, \mathbf{0})$ for all $\beta_1 \in [0,1]$. Furthermore,*

1. *if $s(-I + (I - Z)B^3) < 0$, then the equilibrium set $(\beta_1 z, (1-\beta_1)z, \mathbf{0})$, with $\beta_1 \in [0,1]$, is locally exponentially attractive.*
2. *if $s(-I + (I - Z)B^3) > 0$, then the equilibrium set $(\beta_1 z, (1-\beta_1)z, \mathbf{0})$, with $\beta_1 \in [0,1]$, is unstable.*

[a] S. Gracy, M. Ye, BDO Anderson, C Uribe, ACC'23.

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---

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Proof Sketch:

1. Note that the Jacobian is block upper triangular.

$$J(\beta_1 z, (1-\beta_1)z, \mathbf{0}) =$$

$$\begin{bmatrix} -I + (I-Z)B^1 - \mathrm{diag}(B^1\beta_1 z) & -\mathrm{diag}(B^1\beta_1 z) & -\mathrm{diag}(B^1\beta_1 z) \\ -\mathrm{diag}(B^2(1-\beta_1)z) & -I + (I-Z)B^2 - \mathrm{diag}(B^2(1-\beta_1)z) & -\mathrm{diag}(B^2(1-\beta_1)z) \\ \mathbf{0} & \mathbf{0} & -I + (I-Z)B^3 \end{bmatrix}$$

2. Find a coordinate transform such that the 11-block is irreducible Metzler.
3. Use Center-Manifold theorem.

Global convergence to a plane of coexistence equilibria

We suppose that

1. All three viruses are spreading over the same graph.
2. For all $i \in [n]$ $\delta_i^1 = \delta_i^2 = \delta_i^3 > 0$.
3. For all $i = j \in [n]$ and $(i, j) \in \mathcal{E}$, $\beta_{ij}^1 = \beta_{ij}^2 = \beta_{ij}^3$.

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Theorem

[a] *Consider tri-virus system. Further, suppose that* $\rho(D^{-1}B) > 1$. *Then*

1. *For all initial conditions satisfying* $x^1(0) > \mathbf{0}_n$, $x^2(0) > \mathbf{0}_n$, *and* $x^3(0) > \mathbf{0}_n$, *we have that* $(x^1(t), x^2(t), x^3(t)) \in \mathcal{E}$ *exponentially fast, where*

$$\mathcal{E} = \{(x^1, x^2, x^3) | \alpha_1 x^1 + \alpha_2 x^2 + \alpha_3 x^3 = \tilde{x}, \sum_{i=1}^3 \alpha_i = 1\},$$

   and $\tilde{x}$ *is the unique endemic equilibrium of the single virus SIS dynamics defined by* (D, B).

2. *Every point on the connected set* $\mathcal{E}$ *is a coexistence equilibrium.*

[a] S. Gracy, M. Ye, BDO Anderson, CA Uribe, SIAM Journal of Applied Dynamical Systems'23, Under Review

Co-operative Viruses

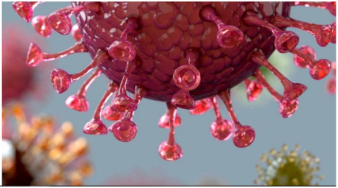
Motivation + model

Covid: Woman aged 90 died with double variant infection

By Michelle Roberts
Health editor, BBC News online

11 July 2021

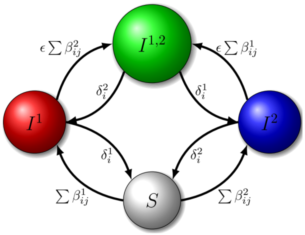
Coronavirus pandemic



Infection with HIV increases the chances of contracting syphilis and vice-versa

Co-operative Viruses

Model



Coupled bivirus SIS model

- $x_i^1(t)$: fraction of individuals in node i infected only with virus 1
- $x_i^2(t)$: fraction of individuals in node i infected only with virus 2
- $z_i(t)$: fraction of individuals in node i infected with both viruses 1 and 2

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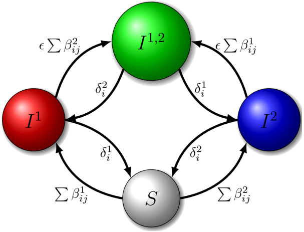

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Equations

$$\begin{aligned}
\dot{x}^1(t) &= -D^1x^1(t) + D^2z(t) + (I - X^1(t) - X^2(t) - Z(t))B^1(x^1(t) + z(t)) \\
&\quad -\epsilon^1X^1(t)B^2(x^2(t) + z(t)), \\
\dot{x}^2(t) &= -D^2x^2(t) + D^1z(t) + (I - X^1(t) - X^2(t) - Z(t))B^2(x^2(t) + z(t)) \\
&\quad -\epsilon^2X^2(t)B^1(x^1(t) + z(t)), \\
\dot{z}(t) &= -(D^1 + D^2)z(t) + \epsilon^1X^1(t)B^2(x^2(t) + z(t)) + \epsilon^2X^2(t)B^1(x^1(t) + z(t)).
\end{aligned}$$

Co-operative viruses

- ϵ^k- the *coupling parameters* between the two viruses.

  Various paradigms
  - If $\epsilon^k \in (0, 1)$, that means if a node is infected with virus k it is less susceptible to the other virus $\rightarrow$ Suppressing regime.
  - If $\epsilon^k > 1$ that means if a node is infected with virus k, it is more likely to become infected with the other virus $\rightarrow$ Reinforcing regime.
- How does it connect to other models that we have seen?
  - If $\epsilon^k = 0$ for $k = 1, 2$, then the co-operative virus model coincides with the competitive bi-virus model.
  - If $\epsilon^k = 1$ for $k = 1, 2$, then the co-operative virus model is the same as two independent single-virus SIS systems.

Co-operative viruses

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