On the existence of equilibria in complex nonlinear networks

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ELLIIT Focus Period
Linköping University

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About me

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  - Hosted by Emma Tegling & Anders Rantzer

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- PhD from University of Groningen (2021)
  - Supervised by Claudio De Persis & Arjan van der Schaft
- Main interests:
  - Power systems
  - Existence of equilibria to nonlinear physical systems
  - (Nonlinear) controller design for vehicle platooning
  - Matrix theory & algebraic graph theory

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- Two parts of this presentation

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  - DC power grids (past work)

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- Two parts of this presentation
  - DC power grids (past work)
  - Hydraulic component of district heating grids (ongoing work)

Part 1

DC power grids and power flow feasibility

- Part I based on two part paper in IEEE Transactions on Automatic Control (Jan 2023)

DC power grids with constant-power loads—Part I: A full characterization of power flow feasibility, long-term voltage stability and their correspondence

Mark Jeeninga*, Claudio De Persis†, Arjan van der Schaft†

DOI 10.1109/TAC.2022.3157076

DC power grids with constant-power loads—Part II: Nonnegative power demands, conditions for feasibility, and high-voltage solutions

Mark Jeeninga*, Claudio De Persis†, Arjan van der Schaft†

DOI 10.1109/TAC.2022.3176808

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  existence of solutions, non-uniqueness of solutions, finding desirable solutions.

Power flow feasibility - approaches

- Classical problem: Research since 1960's
- Convex relaxations of PF equations
- Approximations and simplifications of PF equations (many flavors)
  - DC current flow ("DC power flow approximation")
  - Active-reactive decoupling
  - DistFlow
  - ...
- Excellent survey of Molzahn & Hiskens (2019)
- In general: A fundamental understanding of the PF equations is lacking

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  - Matveev et al. (2020) (DC power grids)
- Stability and high-voltage solutions? Only partial answers.

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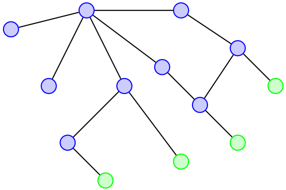
- Applications with constant-power components
  - ships, aircraft, spacecraft, ...
  - DC microgrids and smart grids
  - High-voltage direct current (HVDC) lines (grids?)

The DC power grid model



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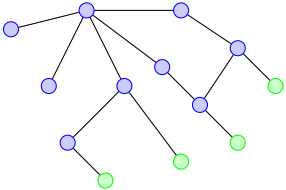
DC power grid at steady state ⇒ model as resistive circuit



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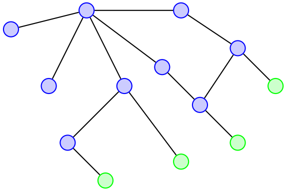

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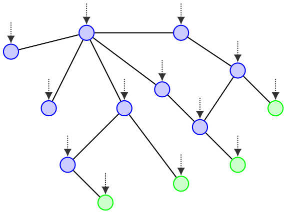

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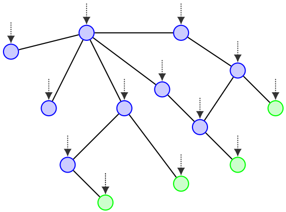
Kirchhoff matrix $Y \in \mathbb{R}^{(n+m)\times(n+m)}$:

$$Y_{ij} := \begin{cases} \sum_{k\sim i} G_{ik} & \text{if } i = j \\ -G_{ij} & \text{if } i \neq j \text{ and } i \sim j \\ 0 & \text{if } i \neq j \text{ and } i \not\sim j \end{cases}.$$

with line conductances $G_{ij} > 0$



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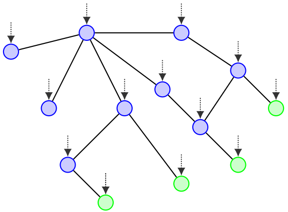
$$V = \begin{pmatrix} V_{\mathrm{L}} \\ V_{\mathrm{S}} \end{pmatrix} > \mathbb{0}$$

Voltage potentials



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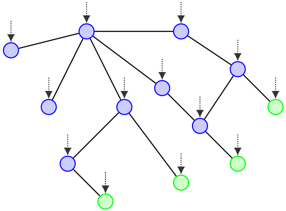

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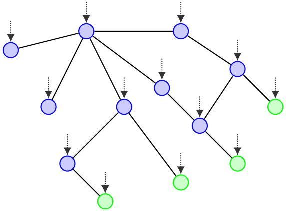
$$\mathcal{I} = \begin{pmatrix} \mathcal{I}_L \\ \mathcal{I}_S \end{pmatrix} = \begin{pmatrix} Y_{\mathrm{LL}} & Y_{\mathrm{LS}} \\ Y_{\mathrm{SL}} & Y_{\mathrm{SS}} \end{pmatrix} \begin{pmatrix} V_{\mathrm{L}} \\ V_{\mathrm{S}} \end{pmatrix}$$

Currents flowing into network at nodes

The DC power grid model



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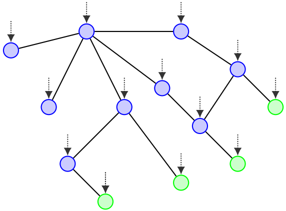
Injected power at the nodes: $P = [V]\mathcal{I}$ $([x] := \mathrm{diag}(x_1, \ldots, x_n))$



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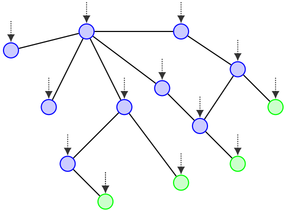

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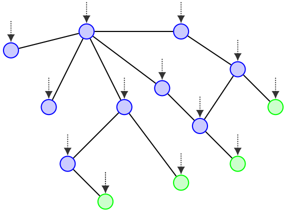
$$V_{\mathrm{L}}^* := -{Y_{\mathrm{LL}}}^{-1} Y_{\mathrm{LS}} V_{\mathrm{S}}$$



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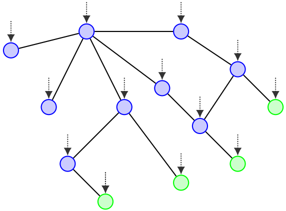

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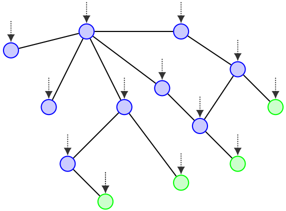

$P_L = [V_{\mathrm{L}}] Y_{\mathrm{LL}} (V_{\mathrm{L}} - V_{\mathrm{L}}^*)$ Power flow feasibility problem: *For which constant power demands* $P_{\mathrm{c}} \in \mathbb{R}^n$ *does there exists an* $operating\ point$ $V_{\mathrm{L}} > 0$ *so that*

$$P_{\mathrm{c}} = -P_L = [V_{\mathrm{L}}] Y_{\mathrm{LL}} (V_{\mathrm{L}}^* - V_{\mathrm{L}}).$$

Inverting the problem: voltage-power correspondence

$$[V_\mathrm{L}]Y_\mathrm{LL}(V_\mathrm{L} - V_\mathrm{L}^*) + P_\mathrm{c} = \mathbb{0}$$

Common approaches:

- Root finding
- Fixed-point methods

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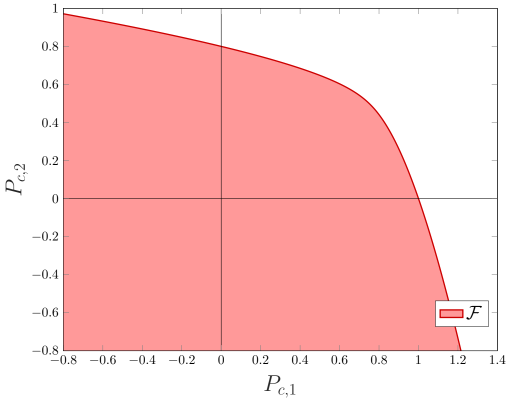
Note: feasibility is a property of all system parameters, not only demands

Set of feasible power demands - example



Jacobian of the voltage-power map

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the long-term voltage stable operating points.

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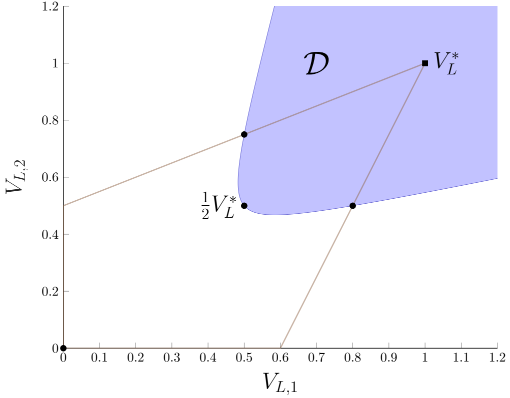
Capture the "increasing load $\Rightarrow$ decreasing steady state voltages"-phenomenon



1.2
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
1.2
$V_{L,1}$
$V_{L,2}$
$\mathcal{D}$
V_L^*
$\frac{1}{2}V_L^*$

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- The following are equivalent:
  - $\frac{\partial P_{\mathrm{c}}}{\partial V_{\mathrm{L}}}(V_{\mathrm{L}})$ is Hurwitz stable;
  - $\frac{\partial P_{\mathrm{c}}}{\partial V_{\mathrm{L}}}(V_{\mathrm{L}})^{-1}$ is a negative matrix;
  - $-\frac{\partial P_{\mathrm{c}}}{\partial V_{\mathrm{L}}}(V_{\mathrm{L}})$ is a nonsingular M-matrix.

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  - $\frac{\partial P_{\mathrm{c}}}{\partial V_{\mathrm{L}}}(V_{\mathrm{L}})^{-1}$ is a negative matrix;
  - $-\frac{\partial P_{\mathrm{c}}}{\partial V_{\mathrm{L}}}(V_{\mathrm{L}})$ is a nonsingular M-matrix.

  Hence,

$$\mathcal{D} = \left\{ V_{\mathrm{L}} \,\middle|\, -\frac{\partial P_{\mathrm{c}}}{\partial V_{\mathrm{L}}}(V_{\mathrm{L}}) \text{ is a nonsingular M-matrix} \right\}$$

- We can parametrize the boundary of $\mathcal{D}$ by the set

$$\{\, \lambda \in \mathbb{R}^n_{>\mathbb{0}} \mid [\lambda] Y_{\mathrm{LL}} + Y_{\mathrm{LL}} [\lambda] \text{ is pos. def.} \,\}$$

Properties of the Jacobian

It can be shown that

- If $V_{\mathrm{L}} > \mathbb{0}$, then $\frac{\partial P_{\mathrm{c}}}{\partial V_{\mathrm{L}}}(V_{\mathrm{L}})$ is Metzler and irreducible;
- The following are equivalent:
  - $\frac{\partial P_{\mathrm{c}}}{\partial V_{\mathrm{L}}}(V_{\mathrm{L}})$ is Hurwitz stable;
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The convex hull of $\mathcal{F}$

Result of Barabanov et al. (2016):

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Theorem

A power demand satisfies $P_c \in \mathcal{F}$ only if there does not exists $\lambda > \mathbb{0}$ such that

$$\begin{pmatrix} Y_{LL}[\lambda] + [\lambda]Y_{LL} & [\lambda]Y_{LS}V_S \\ (Y_{LS}V_S)^\top[\lambda] & 2\lambda^\top P_c \end{pmatrix} \text{ is positive definite.} \quad (1)$$

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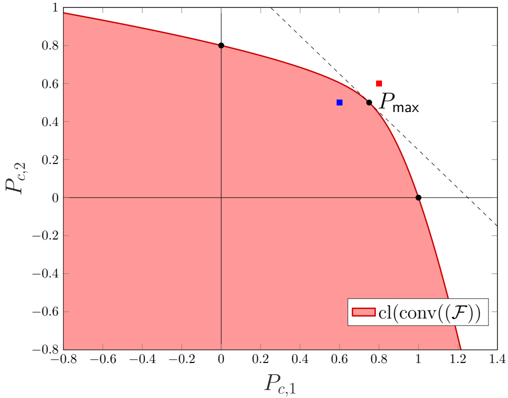
This gives a description of the closure of the convex hull of $\mathcal{F}$.

We show that this is necessary condition is also sufficient.
Consequently, $\mathcal{F}$ is convex and closed.

Necessary and sufficient condition for power flow feasibility - example



$P_{\max}$ is the vector of power demands that maximizes $\sum_i (P_c)_i$.

On the boundary of feasibility

We show that this is necessary condition is also sufficient.

Sketch of the proof:

On the boundary of feasibility

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1. First study the boundary

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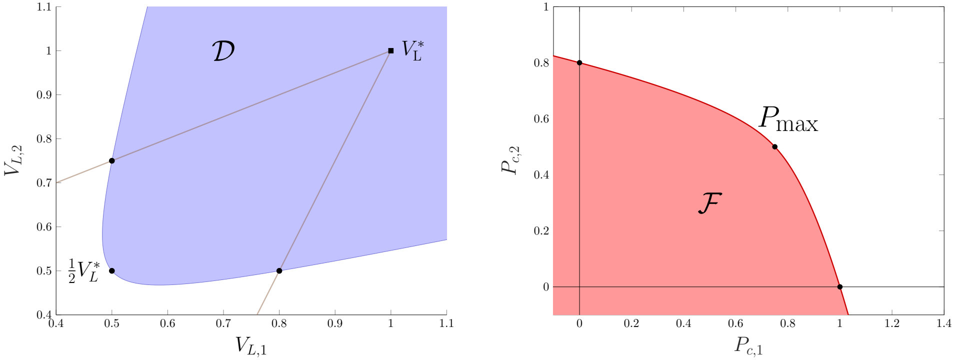
Sketch of the proof:

1. First study the boundary
2. There is a one-to-one correspondence between the boundary of $\mathcal{D}$ and the boundary of $\mathrm{cl}(\mathrm{conv}(\mathcal{F}))$.
3. Show that there are no "holes" or "pockets" in the set $\mathcal{F}$

Finding the operating point - example



$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) = \left(\frac{\partial P_c}{\partial V_L}(\gamma(t))\right)^{-1} \widehat{P}_c$$

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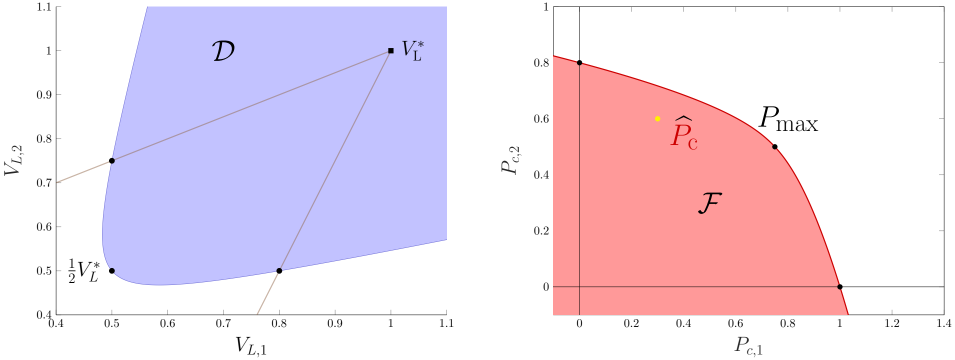

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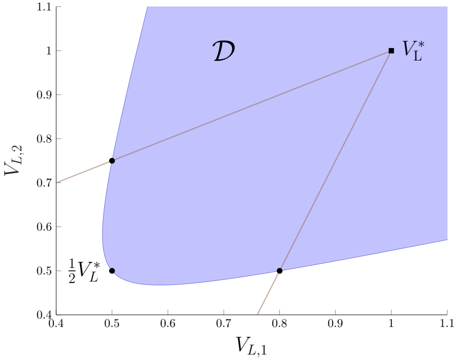

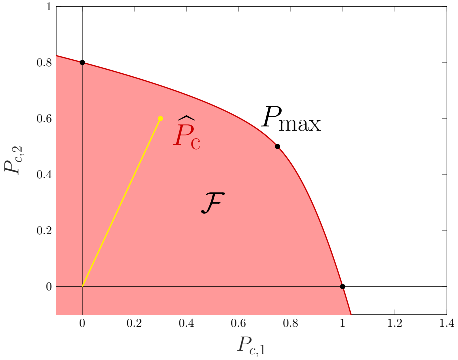

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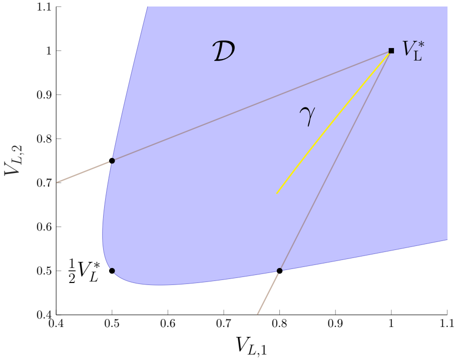

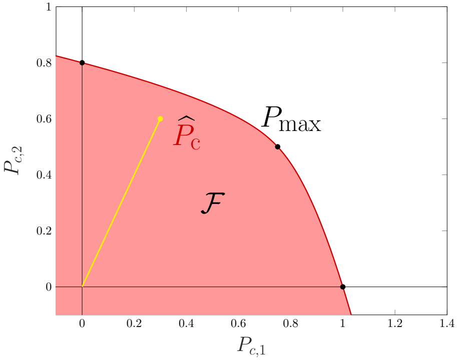

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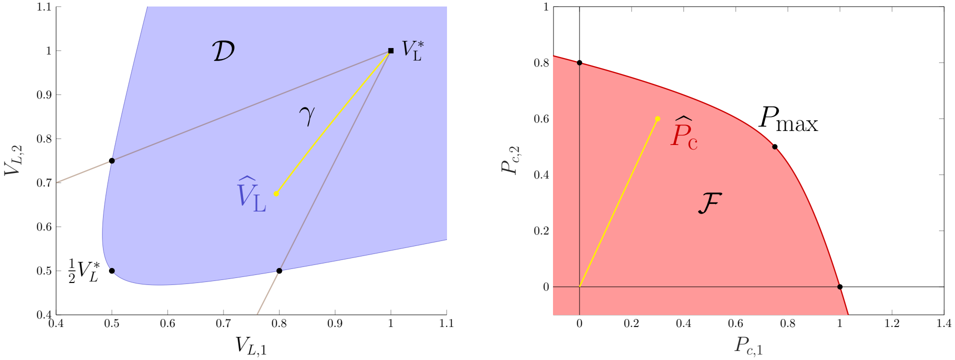

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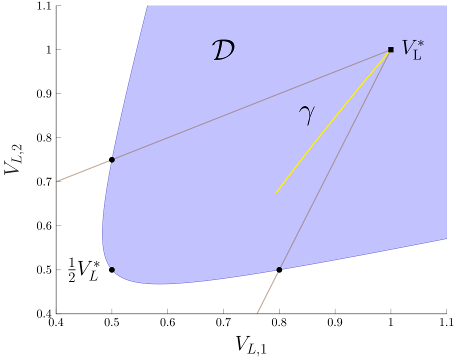

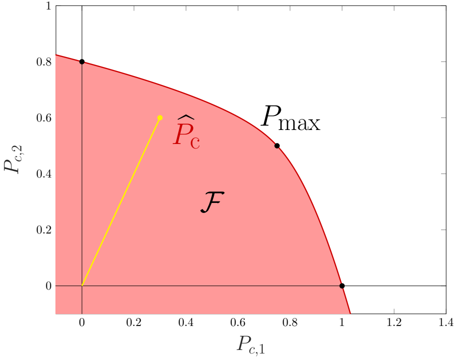

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) = \left(\frac{\partial P_c}{\partial V_L}(\gamma(t))\right)^{-1} \widehat{P}_c$$

γ stays within $\mathcal{D}$ since the boundaries are one-to-one.

One-to-one correspondence for power flow feasibility

Consequences

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- We can find the $\widehat{V}_{\mathrm{L}}$ corresponding to $\widehat{P}_{\mathrm{c}}$ by solving the IVP $\gamma : [0,1] \to \mathbb{R}^n$,

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We have $\gamma(1) = \widehat{V}_{\mathrm{L}}$.

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  We have $\gamma(1) = \widehat{V}_{\mathrm{L}}$.
- The set $\mathcal{F}$ is closed and convex.

Necessary and sufficient condition for power flow feasibility

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Theorem

A power demand satisfies $P_c \in \mathcal{F}$ *if and only if there does not exists* $\lambda > \mathbb{0}$ *such that*

$$Q := \begin{pmatrix} Y_{LL}[\lambda] + [\lambda]Y_{LL} & [\lambda]Y_{LS}V_S \\ (Y_{LS}V_S)^\top[\lambda] & 2\lambda^\top P_c \end{pmatrix}$$

is positive definite.

Similarly, $P_c \in \mathrm{int}(\mathcal{F})$ *if and only if* Q *is never positive semi-definite.*

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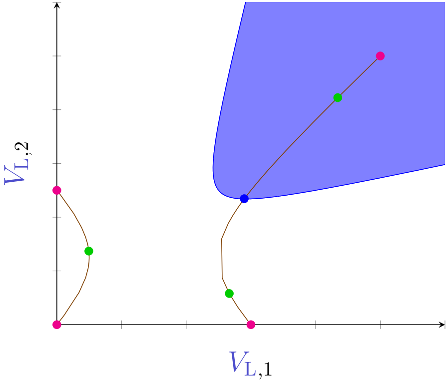
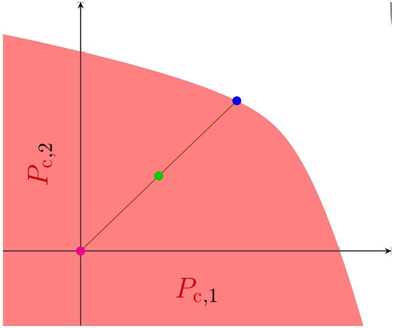
Similarly, $P_c \in \mathrm{int}(\mathcal{F})$ if and only if Q is never positive semi-definite.

Packages like MOSEK solve this problem and its dual, to assess (in)feasibility.

High-voltage solutions and bifurcations





High-voltage solutions and minimizing dissipation

Definition

A high-voltage (HV) solution is an equilibrium $\widehat{V}_{\mathrm{L}}$ such that for all other equilibria V_{L}° we have $(V_{\mathrm{L}}^{\circ})_i < (\widehat{V}_{\mathrm{L}})_i$.

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The HV solution minimizes the dissipation among all equilibria.

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Theorem

If an operating point $\widehat{V}_{\mathrm{L}}$ satisfies one of the following properties, it satisfies all of them:

- $\widehat{V}_{\mathrm{L}}$ *is semi-stable (i.e.,* $\widehat{V}_{\mathrm{L}} \in \mathrm{cl}(\mathcal{D})$*);*
- $\widehat{V}_{\mathrm{L}}$ *is the unique long-term voltage semi-stable operating point associated to* $\widehat{P}_{\mathrm{c}}$*;*
- $\widehat{V}_{\mathrm{L}}$ **is dissipation-minimizing***;*
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This proves practical wisdom.

Other work and open questions

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- Plug & play certificates
- Braess' paradox: increasing line conductances may lead loss of PF feasibility
- Distance to infeasibility

Other work and open questions

- Plug & play certificates
- Braess' paradox: increasing line conductances may lead loss of PF feasibility
- Distance to infeasibility

- How can we make loads behave such that they guarantee feasibility? (Dynamic pricing games?)
- Which dynamics on the loads lead to (global?) convergence in case of feasibility?

Part 2

Equilibria to district heating systems

Joint work with

- Juan Machado (University of Technology Cottbuss - Senftenberg)
- Michele Cucuzzella (University of Pavia)
- Giacomo Como (Politecnico di Torino - Lund University)
- Jacquelien Scherpen (University of Groningen)

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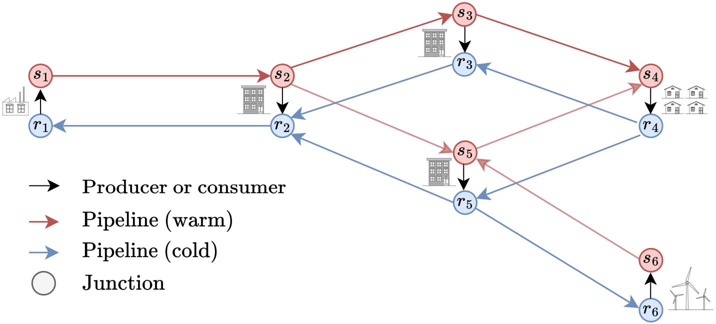
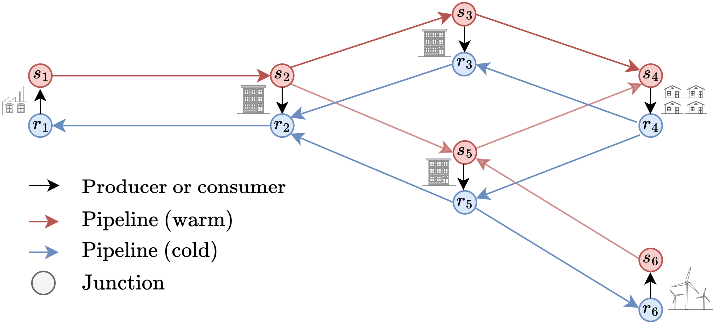
To be presented CDC 2023.

District heating systems

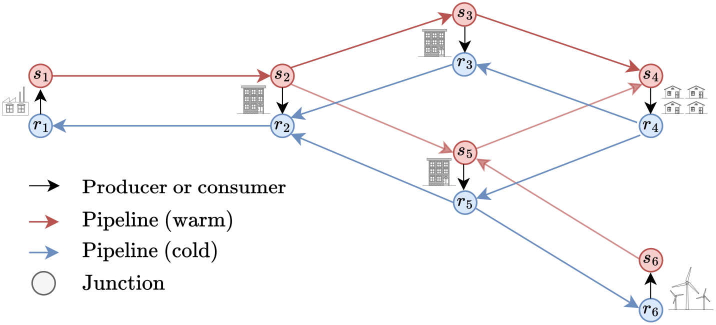


District heating systems



- Transport heat (energy) from producers to consumers

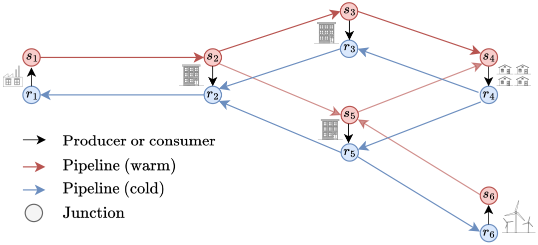
District heating systems



- Transport heat (energy) from producers to consumers
- Cutting costs by reusing waste heat from industrial processes

Components:

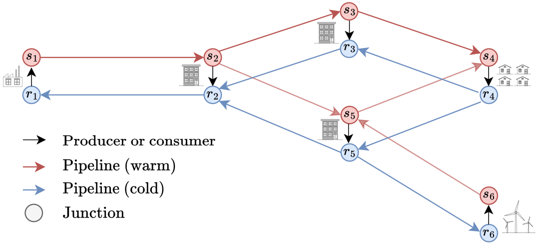
- Pipes



District heating systems

Components:

- Pipes
  - Hot and cold layer



Components:

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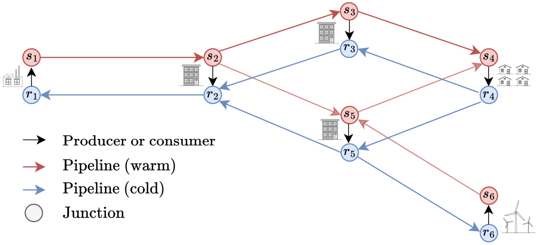
- Heat exchangers



Components:

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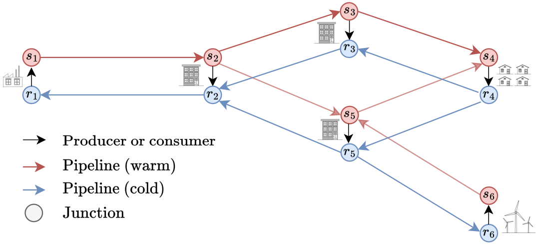
- Pumps, in series with pipes



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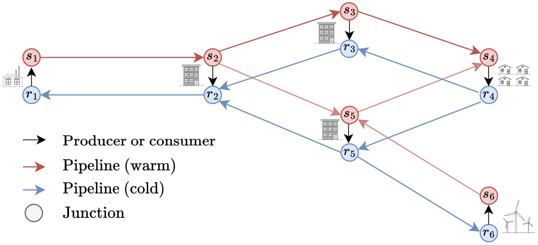
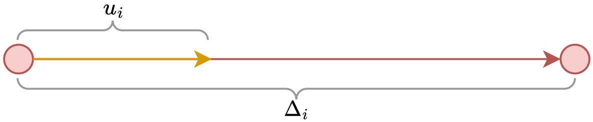


We only focus on these components, but also valves and storage facilities may be considered.

We assume that the fluid is incompressible and laminar flow in the pipes.

Pipe flow dynamics and system model



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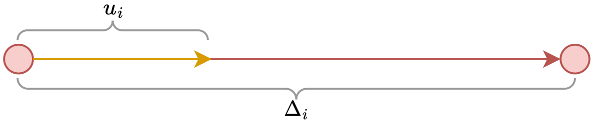

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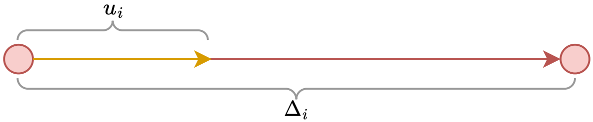

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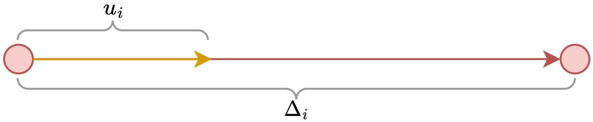

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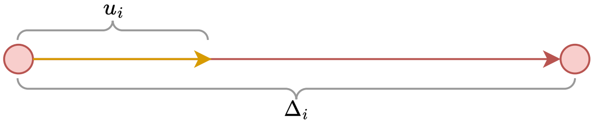

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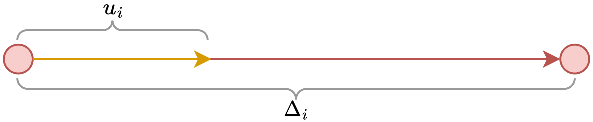

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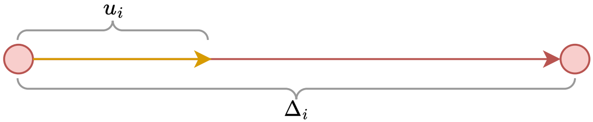

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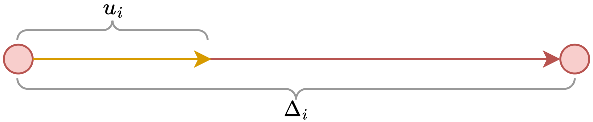
Nonlinear flow in the pipes given by

$$\dot{q}_i = -f_i(q_i) + \Delta_i - u_i$$

Pipe flow dynamics and system model



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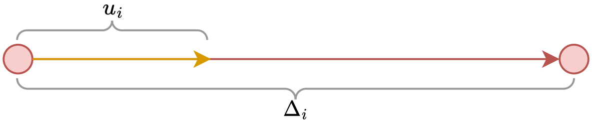

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For hydraulic networks we typically have $f_i(q_i) = -\kappa_i \operatorname{sign}(q_i) q_i^2$.

Control goals

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Control goals

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- Type α: Actuated edge to achieve a flow q_i^* through the edge;

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Control goals

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Control goals

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In particular, at steady state we have

$$u_i = \begin{cases} -f_i(q_i^*) + \Delta_i & \text{if } i \in \alpha \\ -f_i(q_i) + \Delta_i^* & \text{if } i \in \beta \\ 0 & \text{if } i \in \gamma \end{cases}$$

Control goals

Nonlinear flow in the pipes given by

$$\dot{q}_i = -f_i(q_i) + \Delta_i - u_i$$

There are three types of edges:

- Type α: Actuated edge to achieve a flow q_i^* through the edge;
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Assumption: There is no coordination between the actuators of q_i^* and Δ_i^*.

Problem statement

For which placements of the pumps (actuators) does there exist an equilibrium for all choices of q_i^* and Δ_i^*?

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- If $T \subseteq \mathcal{E}$ is a spanning tree, then $k \in \mathcal{E}$ is a twig if $k \in T$, and a chord if $k \in \mathcal{E} \setminus T$.

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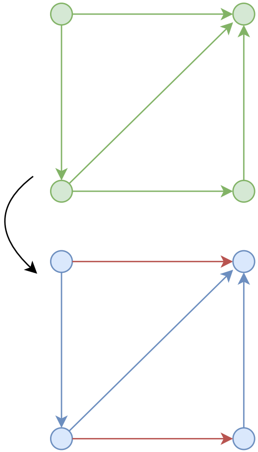

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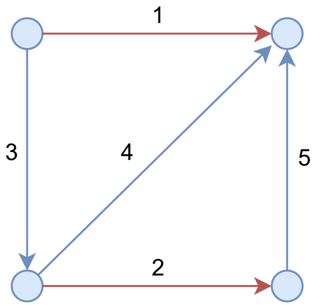
- We have exactly $c = e - n + 1$ chords;
- Adding a chord induces a cycle.

We define $F \in \mathbb{R}^{c \times e}$

$$F_{ki} = \begin{cases} \pm 1 & \text{if } i \text{ lies in the cycle that occurs by adding } k\text{, with sign according to orientation} \\ 0 & \text{if } i \text{ lies in the cycle that occurs by adding } k \end{cases}$$

We have $F = \begin{pmatrix} I & \bigm| & F^* \end{pmatrix}$.

Fundamental loop matrix - example



$$F = \begin{pmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 \end{pmatrix}$$

Main result

$$q \in \operatorname{im} F^\top \qquad \text{(KCL)}$$

$$F\Delta = \mathbb{0} \qquad \text{(KVL)}$$

$$q_i = q_i^* \qquad \text{for } i \in \alpha$$

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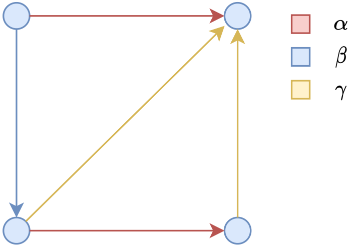

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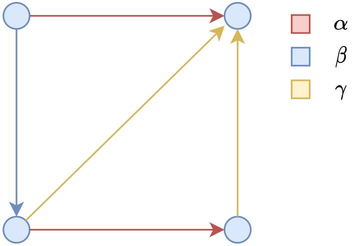

Theorem

There exists an equilibrium $(\widehat{q}, \widehat{\Delta})$ for each choice of q_i^* for $i \in \alpha$ and Δ_i^* for $i \in \beta$ if and only if there exists a subset of edges $T \subseteq \mathcal{E}$, such that T forms a spanning tree and $\beta \subseteq T$ and $\alpha \subseteq \mathcal{E} \setminus T$. Moreover, $(\widehat{q}, \widehat{\Delta})$ is unique.

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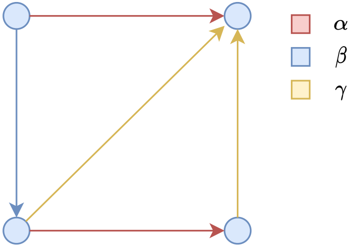

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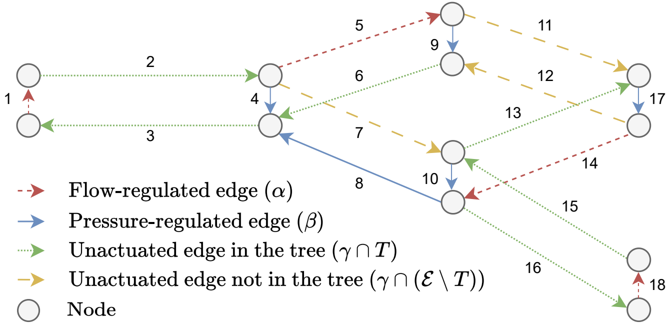
Corollary

The unique equilibrium can be obtained from the solution of a dynamical system $\dot{x} = g(x)$ *where* g *is a monotone bijection.*

Main result - example



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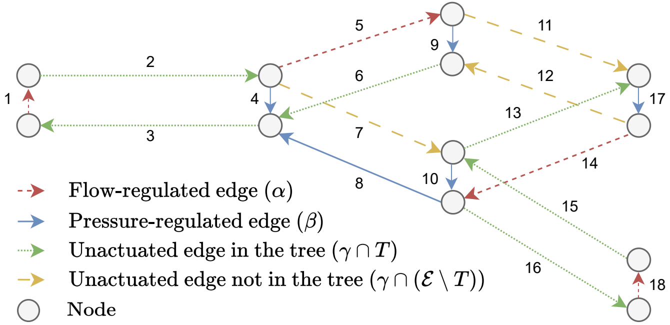

Perhaps unsurprisingly, this coincides with simultaneous cut set and loop analysis.

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- The problem we are solving is much more general. To what other types of systems is this actuator placement problem of interest?
- What changes if we also include the thermal layer into the system of equations?
- What if not all maps f_i are surjective?
- And what if we include constant-power loads?

Thank you for your attention!