Fast and Flexible Decision-Making in Network Systems

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Real World Requires Decision-Making that is Fast and Flexible

Fast

- if it breaks indecision as quickly as indecision becomes costly
- requires fast divergence away from indecision in addition to fast convergence to a decision

Flexible

- if it adapts to signals important to successful operation, even if weak or rare
- requires distinguishing these from unimportant fluctuations *and*

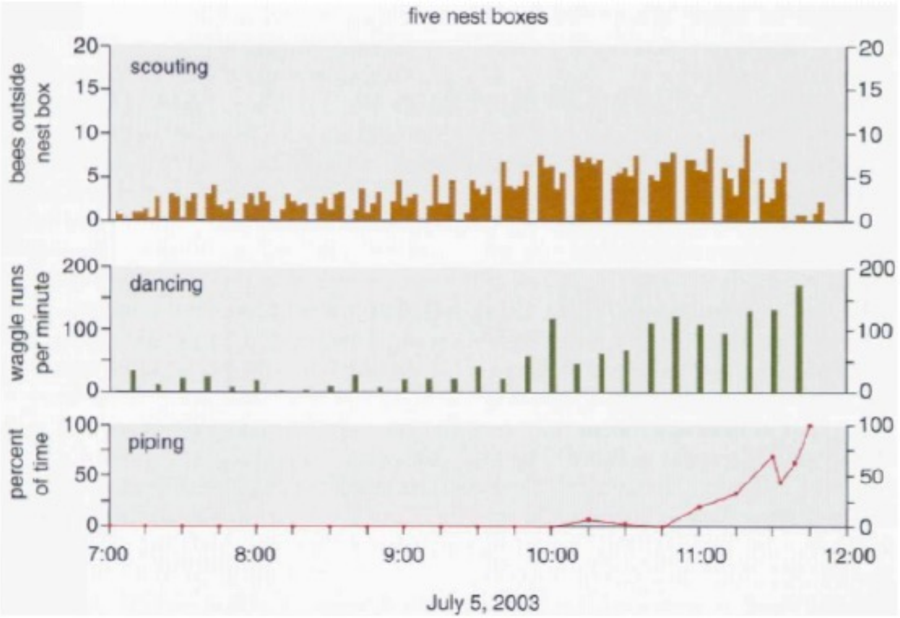
➔ *Tunable sensitivity to inputs*: parameters for modulating

- regimes in which system is ultrasensitive (for flexibility)
- regimes in which the system is insensitive (for robustness)

Essential: ***nonlinearity*** *and* ***feedback*** *in the dynamics with* ***analytical tractability***

Fast and Flexible Decision-Making in Natural Networks

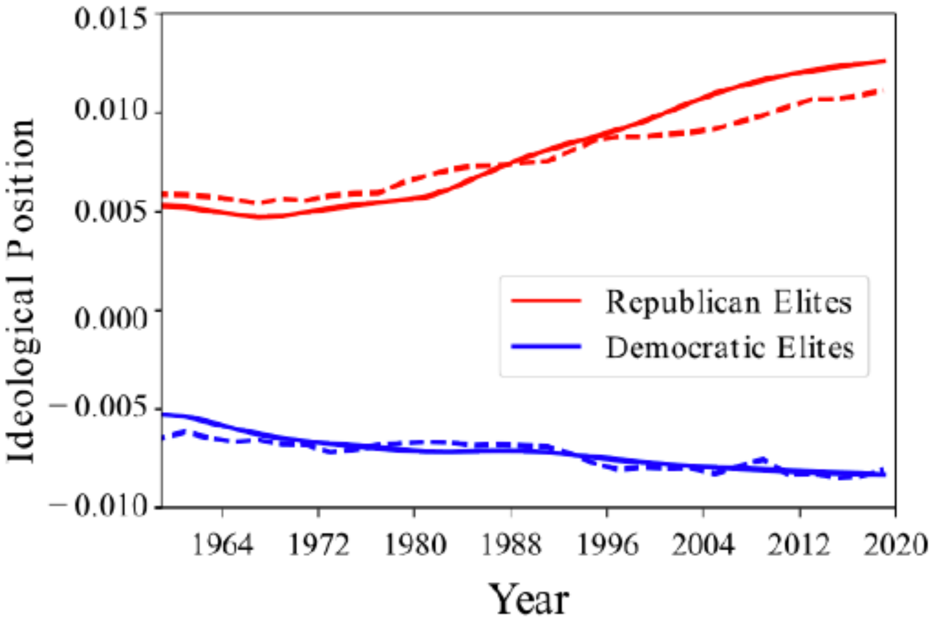
Honey bee swarm
Nest-site selection



Seeley, Visscher, Passino, *American Scientist,* 2006

Pais, Hogan, Schlegel, Franks, Leonard, Marshall, "A mechanism for value-sensitive decision making," *PLoS One,* 2013

US Congress
Political polarization

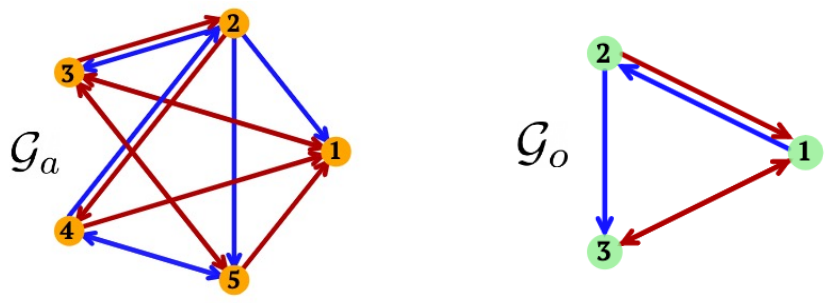


Leonard, Lipsitz, Bizyaeva, Franci, Lelkes, "The nonlinear feedback dynamics of asymmetric political polarization," *PNAS,* 2021

Decision-making on Network Systems

Set of N_a agents: $\mathcal{V}_a = \{1, \ldots, N_a\}$ and set of N_o options: $\mathcal{V}_o = \{1, \ldots, N_o\}$

- communication network $\mathcal{G}_a = (\mathcal{V}_a, \mathcal{E}_a, A_a)$, $A_a = [a^a_{ik}] \in \mathbb{R}^{N_a \times N_a}$
- belief system network $\mathcal{G}_o = (\mathcal{V}_o, \mathcal{E}_o, A_o)$, $A_o = [a^o_{jl}] \in \mathbb{R}^{N_o \times N_o}$



Decision-making Modeled as Dynamical Nonlinear Process*

Dynamical process is coupled evolution over time of agents' decision states:

- agent i's opinion $z_{ij} \in \mathbb{R}$ of each option j. The more positive (negative) is z_{ij}, the more agent i favors (disfavors) option j. When $z_{ij} = 0$, agent i is neutral or undecided about j.

$\boldsymbol{z}$ is opinion vector

- agent i's attention $u_i \geq 0$, which is gain on agent's observations of opinions

$\boldsymbol{u}$ is attention vector

$$attention\ network\ graph\ \mathcal{G}_u = (\mathcal{V}_a, \mathcal{E}_u, A_u), \quad A_u = [a_{ik}^u] \in \mathbb{R}^{N_a \times N_a}$$

Process is organized by **bifurcations**, controlled by **feedback**, and modulated by **network structure**

*We focus on the evolution of opinions and attention in continuous time, but a parallel story can be derived in discrete time.

Model-Independent Approach

Model-independent approach relies solely on empirically verifiable assumptions to make testable predictions for any model or real-world system that verifies (or only weakly violates) them:

Assumption 1: Opinions evolve continuously in time according to a smooth dynamical system.
Any (apparent) discontinuity in opinion-forming behavior is necessarily caused by bifurcation phenomena.

Assumption 2: Opinion formation is a network phenomenon.
Class of dynamical systems that can describe opinion formation determined by theory of network-admissible dynamical systems

Golubitsky M, Stewart I. 2023. *Dynamics and Bifurcation in Networks: Theory and Applications of Coupled Differential Equations.* SIAM

Using symmetry and equivariant bifurcation theory provides "ground truths" for model building:

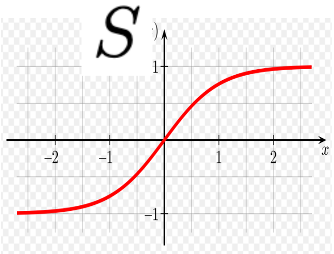
- *E.g., for indistinguishable agents and options, any model of opinion formation should be able to transition from consensus to dissensus through modulation only of the extent of agent cooperativity*

Proposed model

- designed to capture model-independent ground truths
- allows ***model-dependent approaches*** for broader set of contexts

Franci, Golubitsky, Stewart, Bizyaeva, and Leonard, "Breaking indecision in multiagent multioption dynamics", *SIAM Journal of Applied Dynamical Systems*, vol. 22, no. 3, pp. 1780-1817, 2023

Nonlinear Opinion Dynamics



$$\dot{z}_{ij} = -d_{ij} z_{ij} + S\left(u_i\Big(\alpha_i^j z_{ij} + \sum_{\substack{k=1\\k\neq i}}^{N_a} a_{ik}^a z_{kj} + \sum_{\substack{l\neq j\\l=1}}^{N_o} a_{jl}^o z_{il} + \sum_{\substack{k=1\\k\neq i}}^{N_a} \sum_{\substack{l\neq j\\l=1}}^{N_o} a_{ik}^a a_{jl}^o z_{kl}\Big)\right) + b_{ij} \qquad 1.$$

$$\tau_u \dot{u}_i = -u_i + u_0 + K_u \sum_{j=1}^{N_o} \sum_{k=1}^{N_a} a_{ik}^u (z_{kj})^2 \qquad 2.$$

$d_{ij} > 0$ damping coefficient

$\tau_u \geq 0$ time constant

$u_0 > 0$ basal level of attention

$K_u \geq 0$ feedback gain in attention dynamics

$S : \mathbb{R} \to \mathbb{R}$ bounded saturation function with $S(0) = 0$, $S'(0) = 1$, $S'''(0) \neq 0$

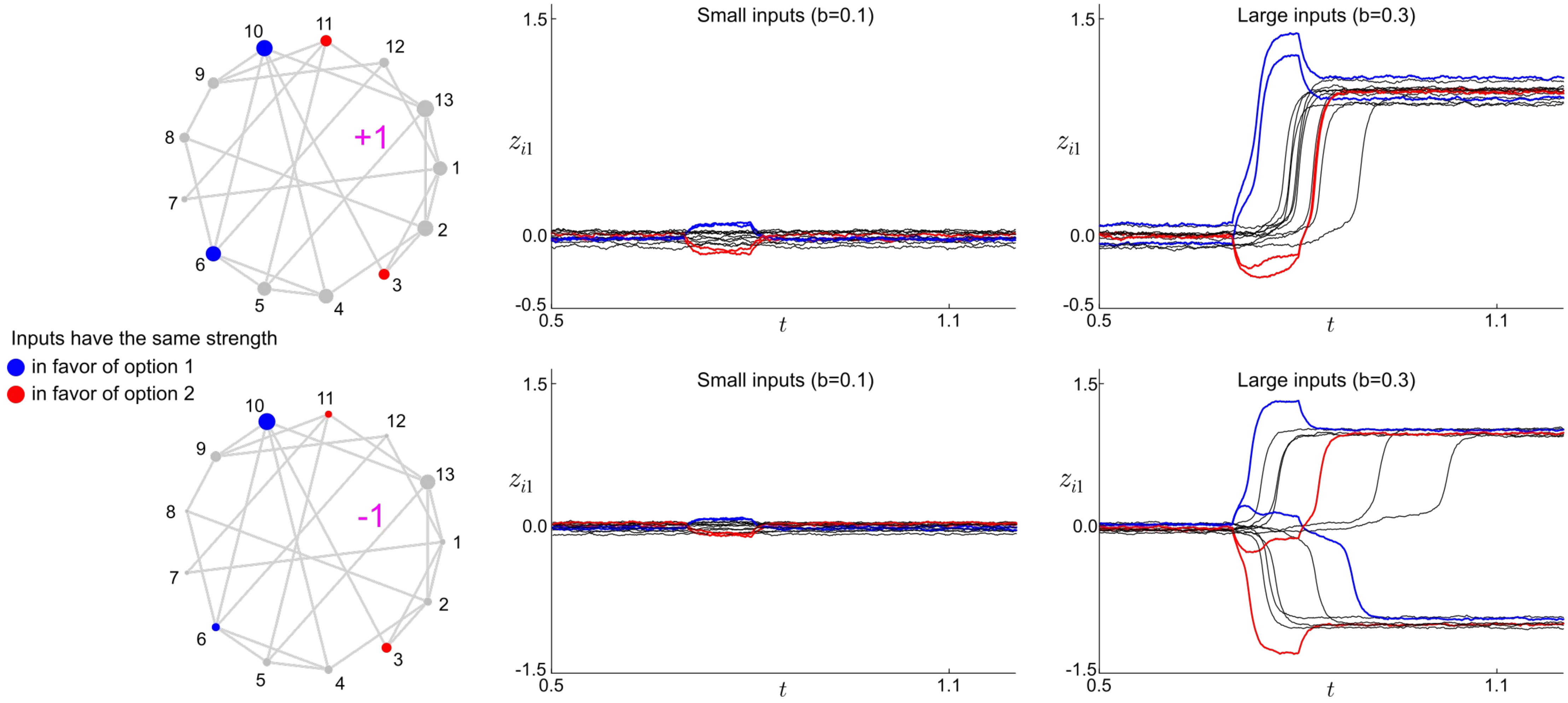
$b_{ij} > 0$ ($b_{ij} < 0$) is input* in favor (disfavor) of option j.

b the input vector

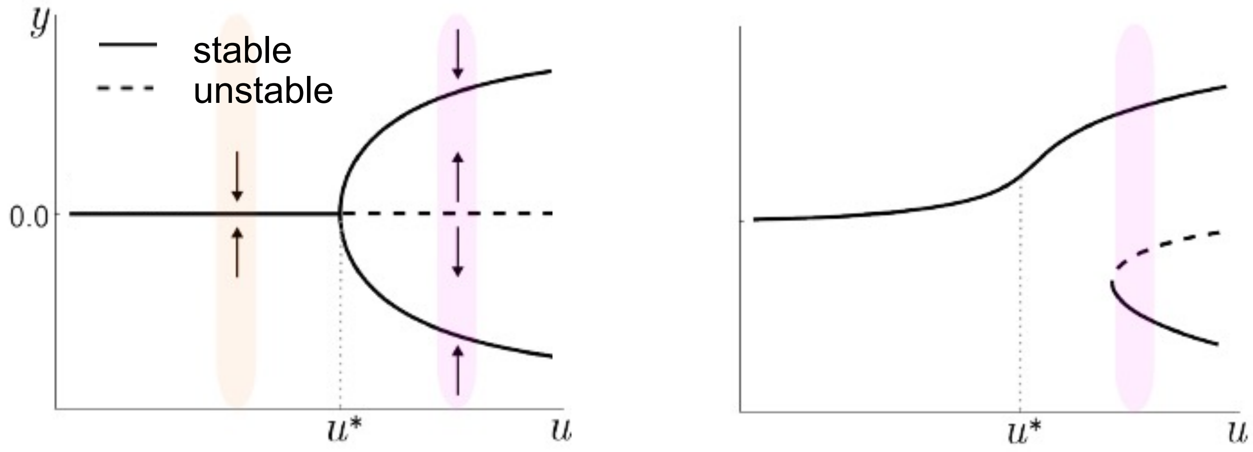
Model can be interpreted as continuous-time recurrent neural network, including finite-dimensional Wilson-Cowan dynamics and continuous Hopfield networks as special cases.

* “input” also includes biases

Opinion exchange and feedback control of attention determine emergence of implicit *distributed network threshold* for formation of strong opinions in response to inputs



Nonlinear Dynamics (Unlike Linear Dynamics) Exhibit Bifurcations



Get ultrasensitivity near bifurcation point because along critical subspace input-output gain blows up

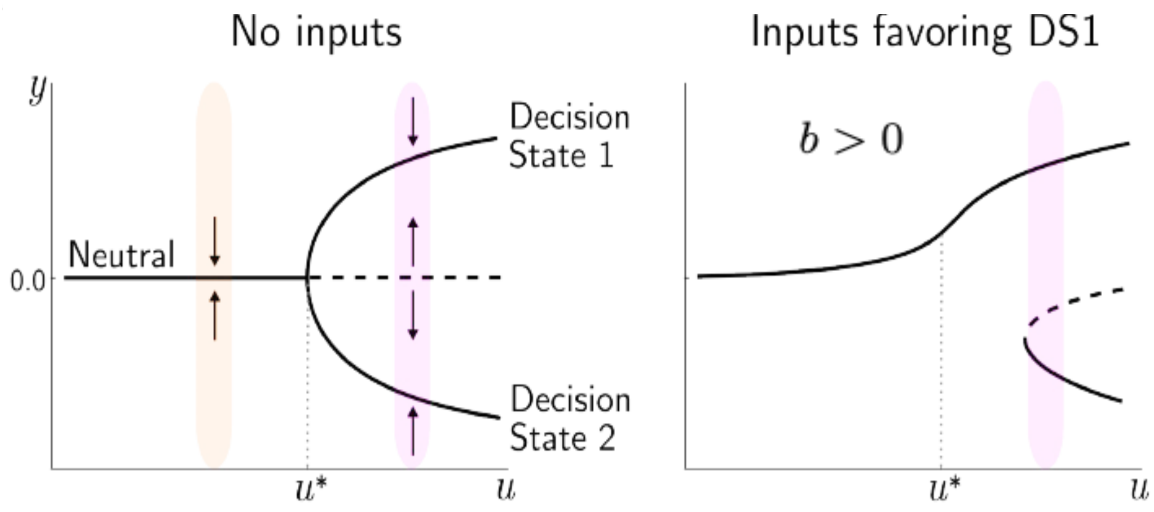
Local bifurcation:
change in number/stability of eq. solutions as (bifurcation) parameter varies across critical value (***bifurcation point***)

At bifurcation point, linearization of dynamics has at least one eigenvalue with zero real part:
- associated eigenspace is the ***critical subspace*** that determines the ***bifurcation center manifold***

Near bifurcation point: process is ***selectively ultrasensitive to input***
- *responsive* to even very small inputs, if inputs excite dynamics along critical subspace
- *robust* to even very large inputs, if inputs do not excite dynamics along critical subspace

Away from bifurcation point: ***multiple stable solutions, robust to small uncertainty***

Opinion Dynamics of Network Systems and Indecision-Breaking Bifurcation



Average attention u is a bifurcation parameter with critical bifurcation value u^*

Critical subspace of $J(\mathbf{0}, u^*)$ is the span of its *right null eigenvector* $\mathbf{v}$ so $y = \langle \mathbf{v}, \boldsymbol{z} \rangle$

Sensitivity subspace of $J(\mathbf{0}, u^*)$ is the span of its *left null eigenvector* $\mathbf{w}$ so $b = \langle \mathbf{w}, \boldsymbol{b} \rangle$

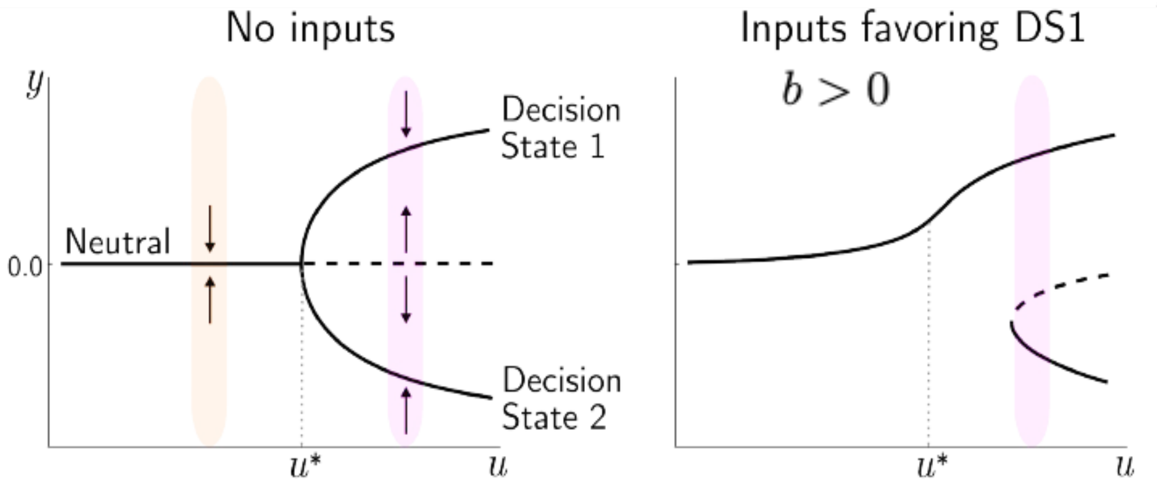
This is proved using ***Lyapunov-Schmidt reduction*** and ***unfolding theory techniques*** (Golubitsky&Schaeffer, 1985)

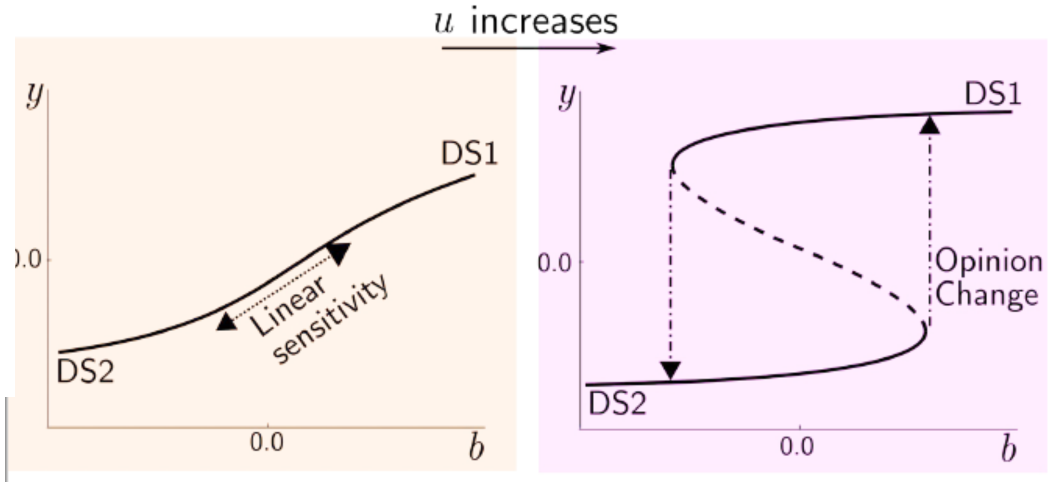
Critical subspace and sensitivity subspace described by eigenstructure of A_a, A_o or $A_a \otimes A_o$

So can prove the role of network structure in decision-making behavior

Opinion Dynamics of Network Systems and Indecision-Breaking Bifurcation

$y = \langle \mathbf{v}, \boldsymbol{z} \rangle$





$b = \langle \mathbf{w}, \boldsymbol{b} \rangle$

When ***u < u*,*** linear negative feedback dominates and opinions linearly track inputs

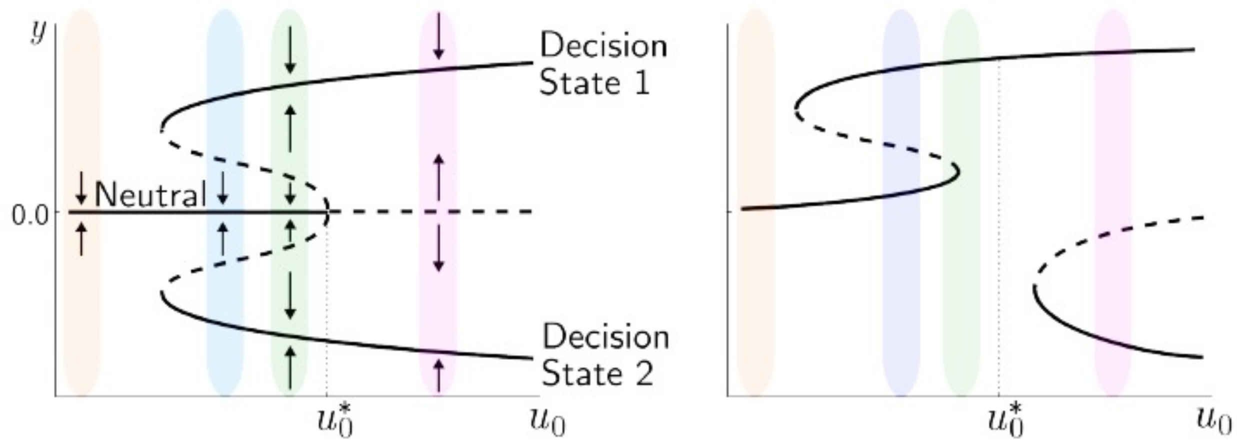
- ***neutral (relatively weak) opinions are stabilized*** for nonexistent (relatively small) inputs

When ***u > u*,*** nonlinear positive feedback dominates and ***strong and robustly stable opinions form***

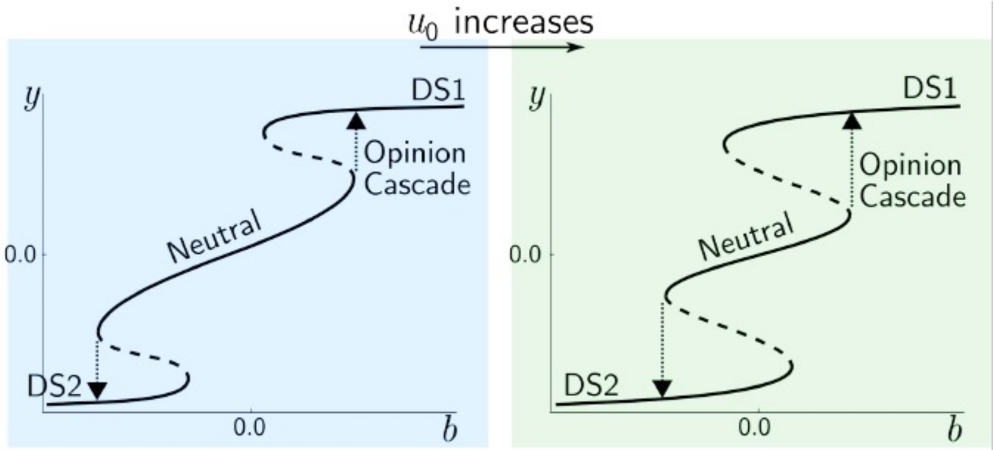
- ***neutral (relatively weak) opinions are destabilized*** even for nonexistent (relatively small) inputs

Opinion Dynamics of Network Systems with State-Dependent Attention

$y = \langle \mathbf{v}, \boldsymbol{z} \rangle$



u_0: basal attention



$b = \langle \mathbf{w}, \boldsymbol{b} \rangle$

Closing the loop between opinion and attention by making attention dependent on opinion state introduces a source of positive feedback that sharpens the pitchfork and can make it subcritical

Transitions between solutions can be very fast and even ***switch-like***

Distributed network threshold tuned by u_0

Opinion Dynamics of Network Systems with State-Dependent Attention: **Two Options**

We first take a close look at the case of two options, $N_o = 2$. If the two options are mutually exclusive, $a^o_{12}, a^o_{21} < 0$ and an opinion in favor of option 1 can be interpreted as an opinion in disfavor of option 2, then we can focus on z_{i1} and let $z_{i2} = -z_{i1}$, for all $i \in \mathcal{V}_a$, Then, the equations for z_{i1} and z_{i2} decouple for every $i \in \mathcal{V}_a$ and Equation 1 becomes

$$\dot{x}_i = -d_i x_i + S\left(u_i(\alpha_i x_i + \sum_{\substack{k=1 \\ k \neq i}}^{N} a_{ik} x_k)\right) + b_i \,, \qquad 3.$$

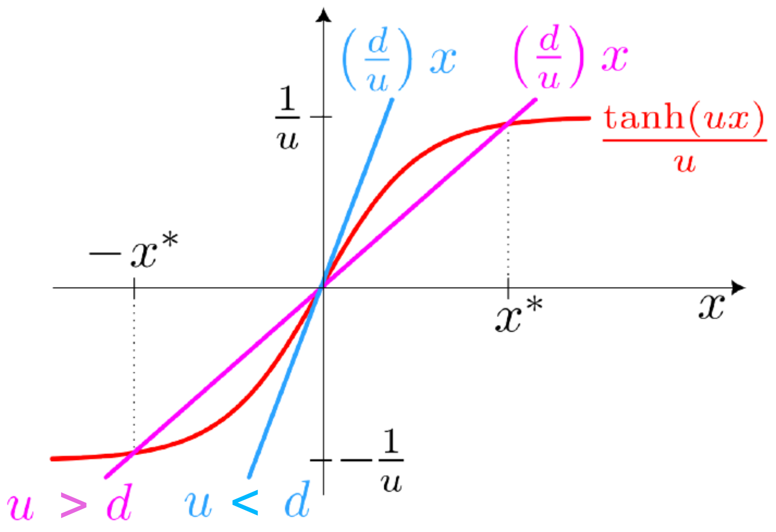
where we have defined $x_i = z_{i1}$, $N_a = N$, $a_{ik} = a^a_{ik} - a^a_{ik} a^o_{12}$, $d_i = d_{i1}$, $\alpha_i = \alpha^1_i - a^o_{12}$, and $b_i = b_{i1}$. We let $A = [a_{ik}]$.

$$\tau_u \dot{u}_i = -u_i + u_0 + K_u \sum_{k=1}^{N} a^u_{ik} x^2_k \,. \qquad 4.$$

Single Agent Opinion Dynamics

$$\dot{x} = -d\,x + \tanh(ux) + b \qquad 5.$$

where $d = d_{i1}$, $u = u_i\alpha_i$, and $b_{i1} = b$. We let $S(\cdot) = \tanh(\cdot)$ without loss of generality.



Equilibria of Eq. 5 for $b = 0$

Single Agent Opinion Dynamics

$$\dot{x} = -d\,x + \tanh(ux) + b \qquad 5.$$

where $d = d_{i1}$, $u = u_i\alpha_i$, and $b_{i1} = b$. We let $S(\cdot) = \tanh(\cdot)$ without loss of generality.

Step 1. *Lyapunov-Schmidt reduction*:

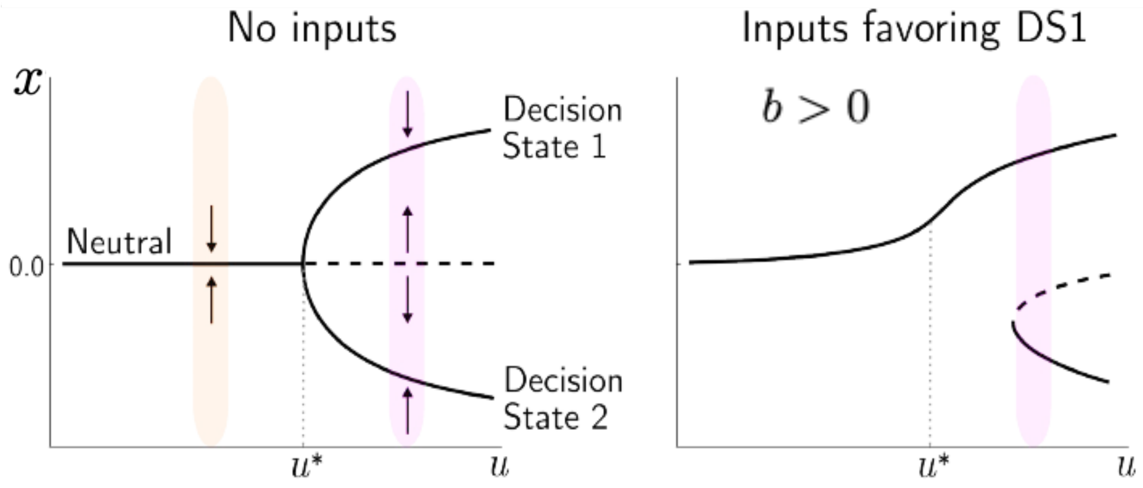
$-dx + \tanh(ux) + b = 0$

Step 2. *Identification*: For $b = 0$, near bif. pt.,

isomorphic to $(u - d)y - (u/3)y^3 = 0$,

normal form for supercritical pitchfork, $u^* = d$

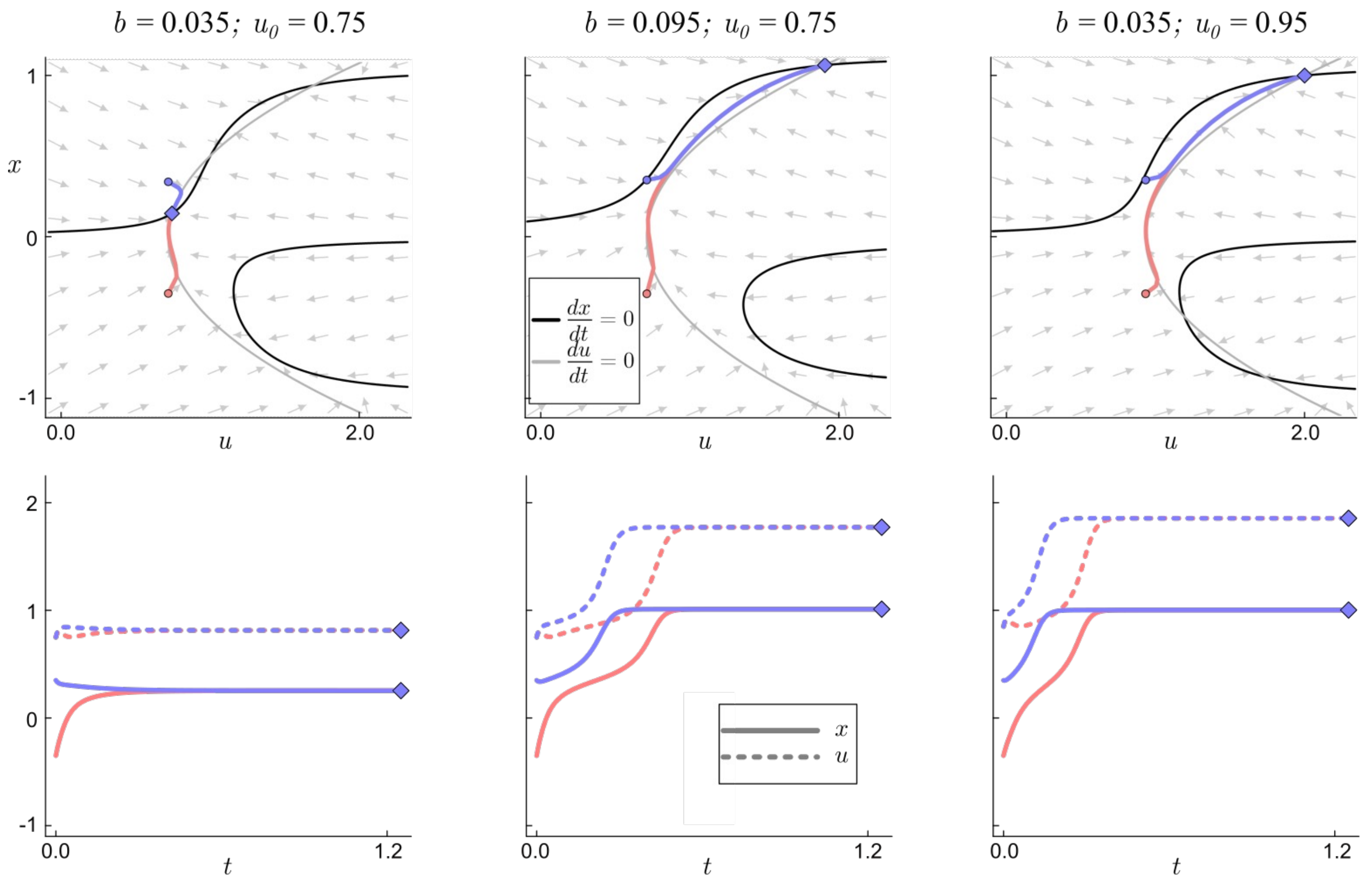
Step 3. *Unfolding theory*



Single Agent Opinion and State-Dependent Attention

$$\dot{x} = -d\,x + \tanh(ux) + b \qquad 5.$$

$$\tau_u \dot{u} = -u + u_0 + K_u x^2 \qquad 6.$$



$d = 1,\ K_u = 1$

Single Agent Opinion and Attention Dynamics

$$\dot{x} = -d\,x + \tanh(ux) + b \qquad 5.$$

$$\tau_u \dot{u} = -u + u_0 + K_u x^2 \qquad 6.$$

Step 1. *Lyapunov-Schmidt reduction*:

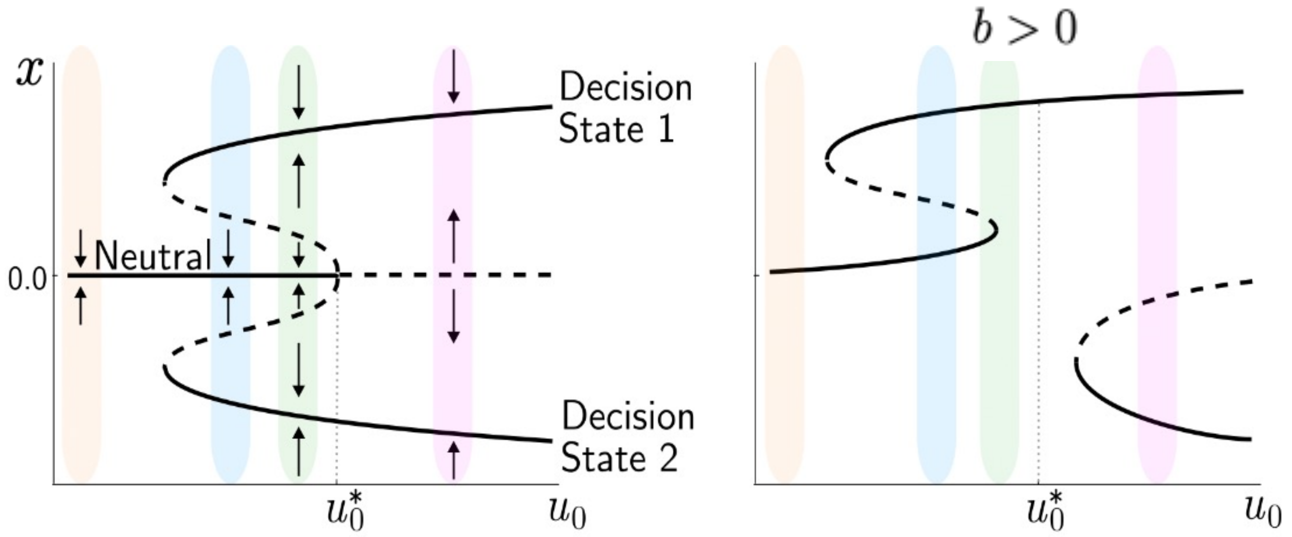
$-d\,x + \tanh((u_0 + K_u x^2)x) + b = 0$

Step 2. *Identification*: For $b = 0$, near bif. pt.,

isomorphic to $(u_0 - d)x - (K_u - u_0/3)x^3 + (K_u/3)x^5$

normal form for "quintic" pitchfork, $u_0^* = d$

Step 3. *Unfolding theory*



Two Agent Opinion Dynamics

$$\dot{x}_1 = -d\,x_1 + \tanh(u(\alpha x_1 + a_{12}x_2)) + b_1$$
$$\dot{x}_2 = -d\,x_2 + \tanh(u(\alpha x_2 + a_{21}x_1)) + b_2\,. \qquad 7.$$

Let $\boldsymbol{b} = \mathbf{0}$. The neutral state $\boldsymbol{x} = \mathbf{0}$ is an equilibrium of Equation 7 for all $u \geq 0$, and the Jacobian evaluated at $\boldsymbol{x} = \mathbf{0}$ is $J = (-d + u\alpha)I + uA$, where A is the adjacency matrix for the two-node network. Let $\lambda_{\max}$ be the eigenvalue of A with largest real part and $\mathbf{v}_{\max}$ and $\mathbf{w}_{\max}$ the corresponding right and left unit eigenvectors. J has largest eigenvalue

$$\lambda_1' = (-d + u\alpha + u\lambda_{\max}) = (\alpha + \lambda_{\max})(u - u^*), \quad u^* = \frac{d}{\alpha + \lambda_{\max}}, \quad \mathbf{v}_1' = \mathbf{v}_{\max}, \quad \mathbf{w}_1' = \mathbf{w}_{\max}$$

We expect a supercritical pitchfork bifurcation at the critical point $\boldsymbol{x} = \mathbf{0}$ and $u = u^*$ with two new stable equilibria appearing for $u > u^*$ along the center manifold, which at $\boldsymbol{x} = \mathbf{0}$ is tangent to *critical subspace* $\mathrm{Ker}(J) = \mathbf{v}_1' = \mathbf{v}_{\max}$

Let $y = \langle \boldsymbol{x}, \mathbf{v}_{\max} \rangle$

Two Agent Opinion Dynamics

$$\dot{x}_1 = -d\,x_1 + \tanh(u(\alpha x_1 + a_{12}x_2)) + b_1$$
$$\dot{x}_2 = -d\,x_2 + \tanh(u(\alpha x_2 + a_{21}x_1)) + b_2\,. \qquad 7.$$

Example: $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \lambda_1 = 1, \mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \lambda_2 = -1, \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$(\lambda_{max}, \mathbf{v}_{\max}, \mathbf{w}_{\max}) = (\lambda_1, \mathbf{v}_1, \mathbf{v}_1)$

Step 1. *Lyapunov-Schmidt reduction:*

$(y_1, y_2) = T^{-1}(x_1, x_2), T = [\mathbf{v}_1, \mathbf{v}_2]$. Rows of T^{-1} are the left eigenvectors $\mathbf{w}_1, \mathbf{w}_2$ of A, so

$y_1 = \langle \boldsymbol{x}, \mathbf{w}_1 \rangle = \langle \boldsymbol{x}, \mathbf{w}_{\max} \rangle = (x_1 + x_2)/\sqrt{2}$ and $y_2 = \langle \boldsymbol{x}, \mathbf{w}_2 \rangle = (x_1 - x_2)/\sqrt{2}$.

$$\dot{y}_1 = -d\,y_1 + (1/\sqrt{2})(\tanh(\tilde{u}(p_s y_1 + p_d y_2)) + \tanh(\tilde{u}(p_s y_1 - p_d y_2)) + b_1 + b_2) \qquad 9.$$
$$\dot{y}_2 = -d\,y_2 + (1/\sqrt{2})(\tanh(\tilde{u}(p_s y_1 + p_d y_2)) - \tanh(\tilde{u}(p_s y_1 - p_d y_2)) + b_1 - b_2)\,. \qquad 10.$$

$p_s = \alpha + 1,$
$p_d = \alpha - 1$
$\tilde{u} = u/\sqrt{2}.$

$\boldsymbol{x} = T\boldsymbol{y} = y_1\mathbf{v}_1 + y_2\mathbf{v}_2$. Set $y_2 = 0$ to restrict to critical subspace Ker(J)= $\mathbf{v}_{\max} = \mathbf{v}_1$: $\quad \dot{y}_1 \approx -d\,y_1 + \tanh(u(\alpha+1)y_1) + \langle \mathbf{w}_1, \boldsymbol{b} \rangle$

Along critical subspace $y = \langle \boldsymbol{x}, \mathbf{v}_{\max} \rangle = y_1 = (x_1 + x_2)/\sqrt{2}$

so reduction is $-d\,y + \tanh(u(\alpha+1)y) + b = 0, \quad b = \langle \mathbf{w}_{\max}, \boldsymbol{b} \rangle = \langle \mathbf{w}_1, \boldsymbol{b} \rangle = (b_1 + b_2)/\sqrt{2}$

Two Agent Opinion Dynamics

$$\dot{x}_1 = -d\,x_1 + \tanh(u(\alpha x_1 + a_{12}x_2)) + b_1$$
$$\dot{x}_2 = -d\,x_2 + \tanh(u(\alpha x_2 + a_{21}x_1)) + b_2\,. \qquad 7.$$

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$(\lambda_{max}, \mathbf{v}_{\max}, \mathbf{w}_{\max}) = (\lambda_1, \mathbf{v}_2, \mathbf{v}_2)$

Step 1. *Lyapunov-Schmidt reduction:*

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$$\dot{y}_1 = -d\,y_1 + (1/\sqrt{2})(\tanh(\tilde{u}(p_s y_1 + p_d y_2)) + \tanh(\tilde{u}(p_s y_1 - p_d y_2)) + b_1 + b_2) \qquad 9.$$
$$\dot{y}_2 = -d\,y_2 + (1/\sqrt{2})(\tanh(\tilde{u}(p_s y_1 + p_d y_2)) - \tanh(\tilde{u}(p_s y_1 - p_d y_2)) + b_1 - b_2)\,. \qquad 10.$$

$p_s = \alpha + 1$,
$p_d = \alpha - 1$
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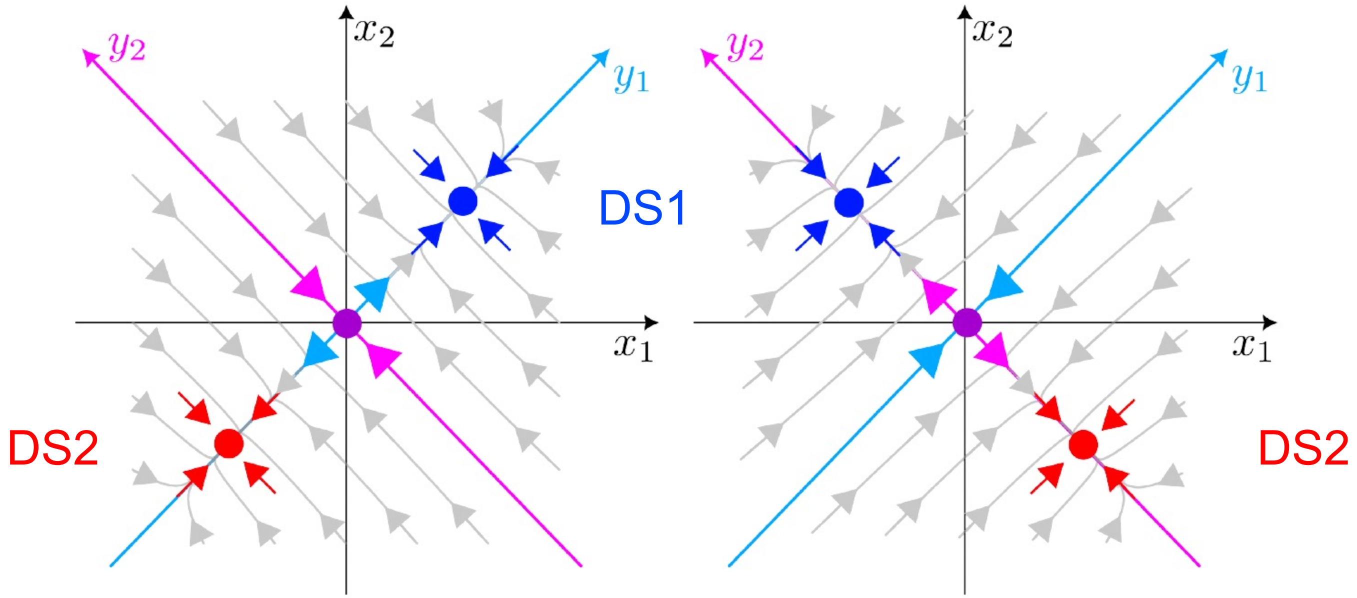
Along critical subspace $y = \langle \boldsymbol{x}, \mathbf{v}_{\max} \rangle = y_2 = (x_1 - x_2)/\sqrt{2}$.

so reduction is $-d\,y + \tanh(u(\alpha+1)y) + b = 0, \quad b = \langle \mathbf{w}_{\max}, \boldsymbol{b} \rangle = \langle \mathbf{w}_2, \boldsymbol{b} \rangle = (b_1 - b_2)/\sqrt{2}$

Two Agent Opinion Dynamics

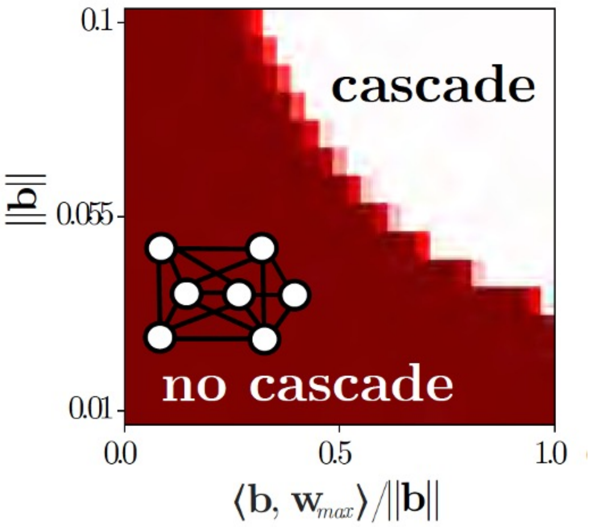
$$u > u^* = \frac{d}{\alpha+1}$$



$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad A = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

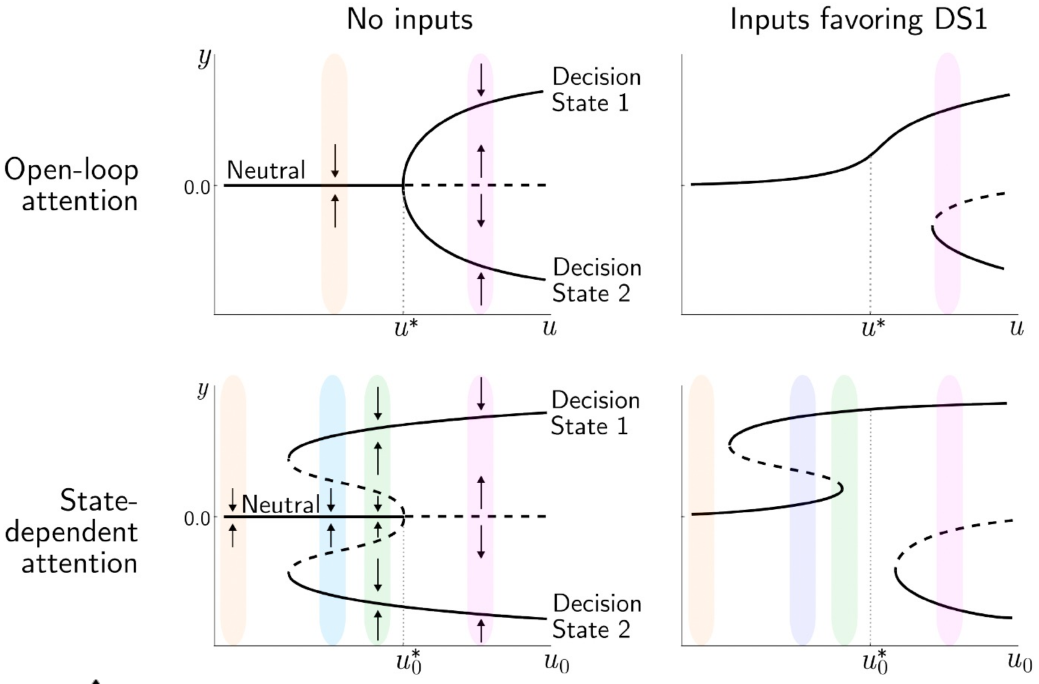
N Agent Opinion Dynamics

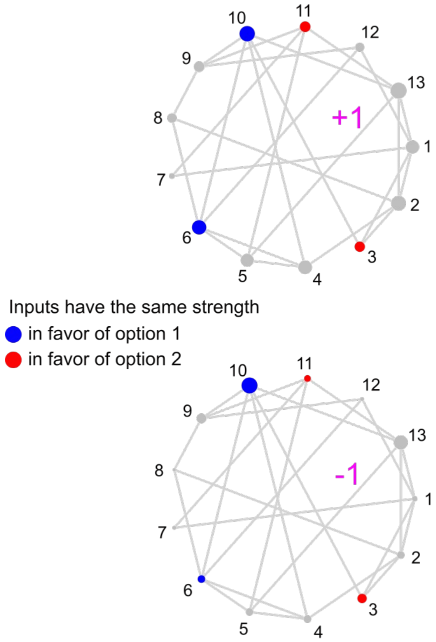
$$y = \langle \boldsymbol{x}, \mathbf{v}_{\max} \rangle \text{ and } b = \langle \mathbf{w}_{\max}, \boldsymbol{b} \rangle$$

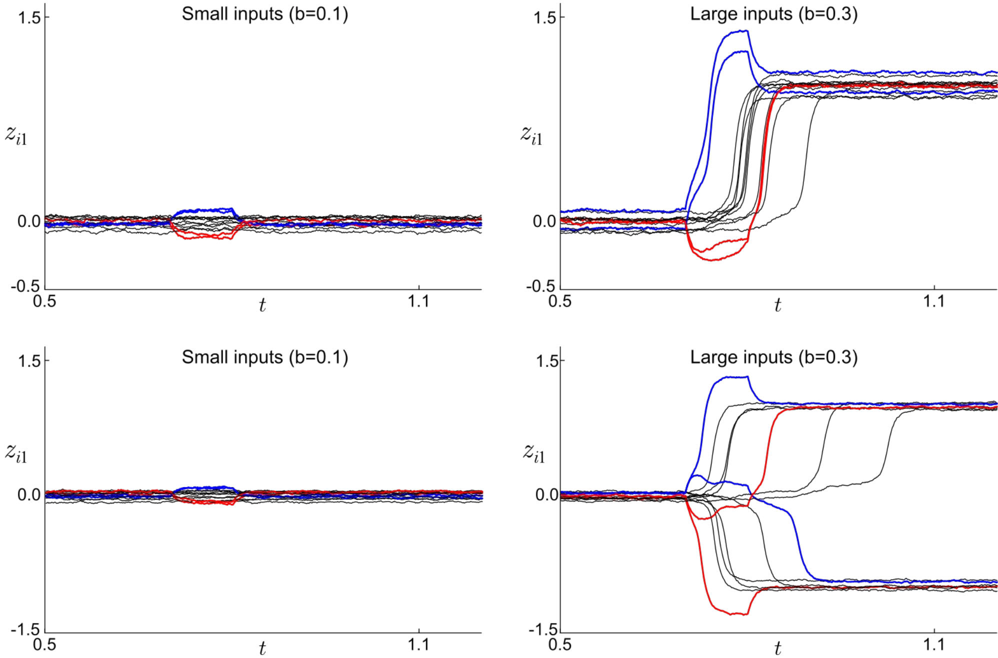


Bizyaeva, Franci, and Leonard, "Nonlinear opinion dynamics with tunable sensitivity", *IEEE Transactions on Automatic Control*, vol. 68, no. 3, pp. 1415-1430, 2023.

Bizyaeva, Amorim, Santos, Franci, and Leonard, "Switching transformations for decentralized control of opinion patterns in signed networks: Application to dynamic task allocation", *IEEE Control Systems Letters*, vol. 6, 2022







N_a Agent and N_o Option Opinion Dynamics

An *indecision-breaking bifurcation* typically happens along either

1) product of leading eigenspaces of A_a, A_o (associated with Λ_1) or

2) leading eigenspace of $A_a \otimes A_o$ (associated with Λ_2)

Λ_1 is set of ordered pairs $(\lambda, \mu) \in \sigma(A_a) \times \sigma(A_o)$ for which $\mathrm{Re}(\lambda) = \lambda_{max}$, $\mathrm{Re}(\mu) = \mu_{max}$

Λ_2 is set of ordered pairs $(\lambda, \mu) \in \sigma(A_a) \times \sigma(A_o)$ for which $\mathrm{Re}(\lambda\mu) = (\lambda\mu)_{max}$

We prove conditions on network graph for kind of bifurcation and predict post-bifurcation behavior:

Consider a signed graph $\mathcal{G}$ on n vertices.

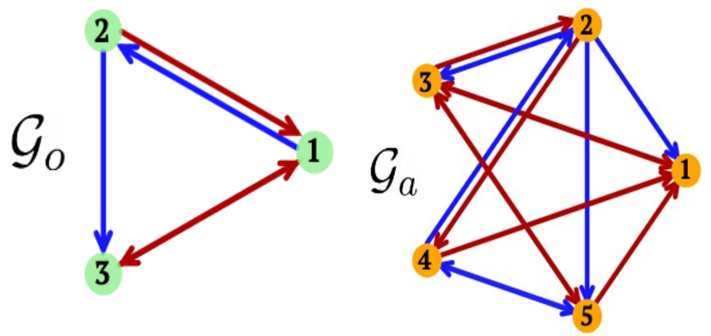
Class I graph $\mathcal{G}$ is switching isomorphic to $\mathcal{G}'$ with eventually positive adjacency matrix A'

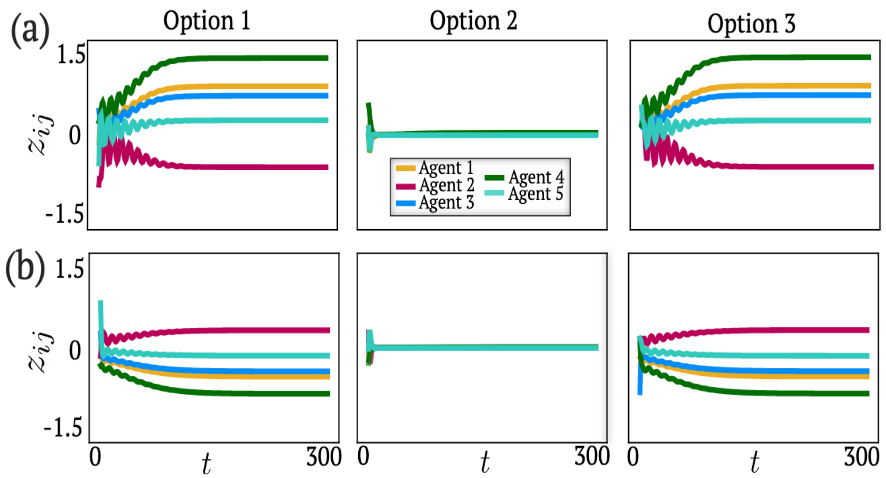
Class II graph $\mathcal{G}$ is digon-symmetric, strongly connected, and structurally balanced

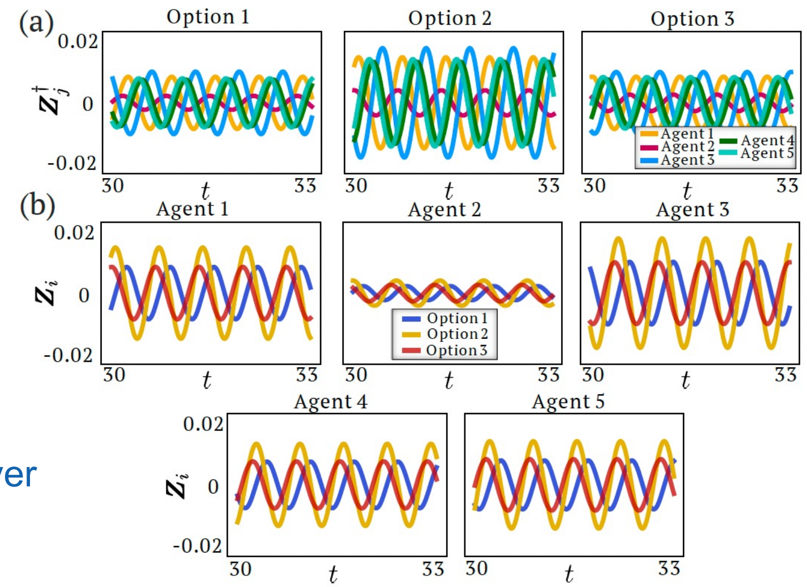
Bizyaeva, Franci, Leonard, Multi-topic belief formation through bifurcations over signed social networks, arXiv:2308.02755 [physics.soc-ph], 2023

N_a Agent and N_o Option Opinion Dynamics

Example: 5 agents and 3 options





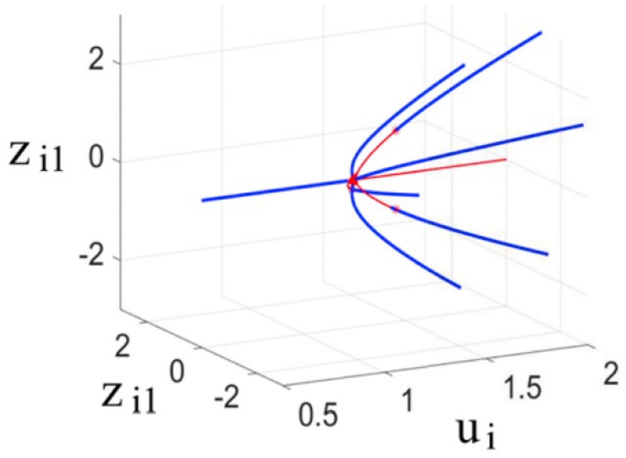


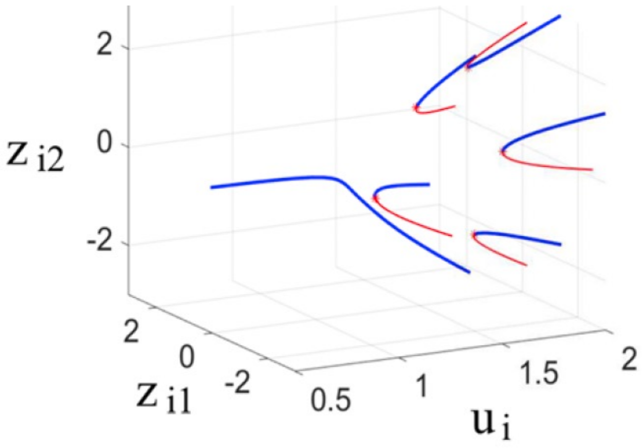
Bizyaeva, Franci, Leonard, Multi-topic belief formation through bifurcations over signed social networks, arXiv:2308.02755 [physics.soc-ph], 2023

N_a Agent and N_o Option Opinion Dynamics

When Options are Indistinguishable

For N_o options, the indecision-breaking bifurcations are predicted by equivariant bifurcation theory for dynamical systems that are equivariant with respect to permuting N_o sets of variables: the agents' opinions about the N_o indistinguishable options. These bifurcations are multi-branch generalizations of the pitchfork.

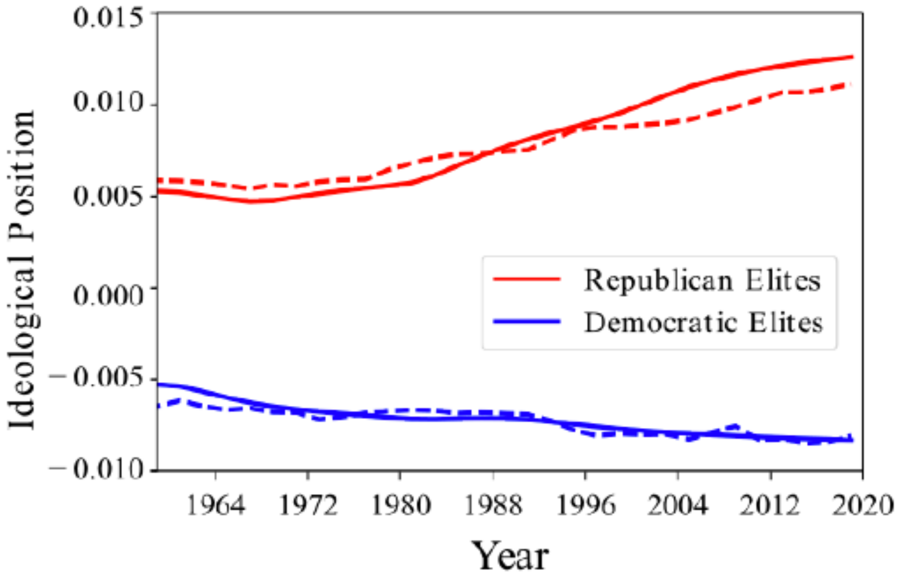




Franci, Golubitsky, Stewart, Bizyaeva, and Leonard, "Breaking indecision in multiagent multioption dynamics", *SIAM Journal of Applied Dynamical Systems*, vol. 22, no. 3, pp. 1780-1817, 2023

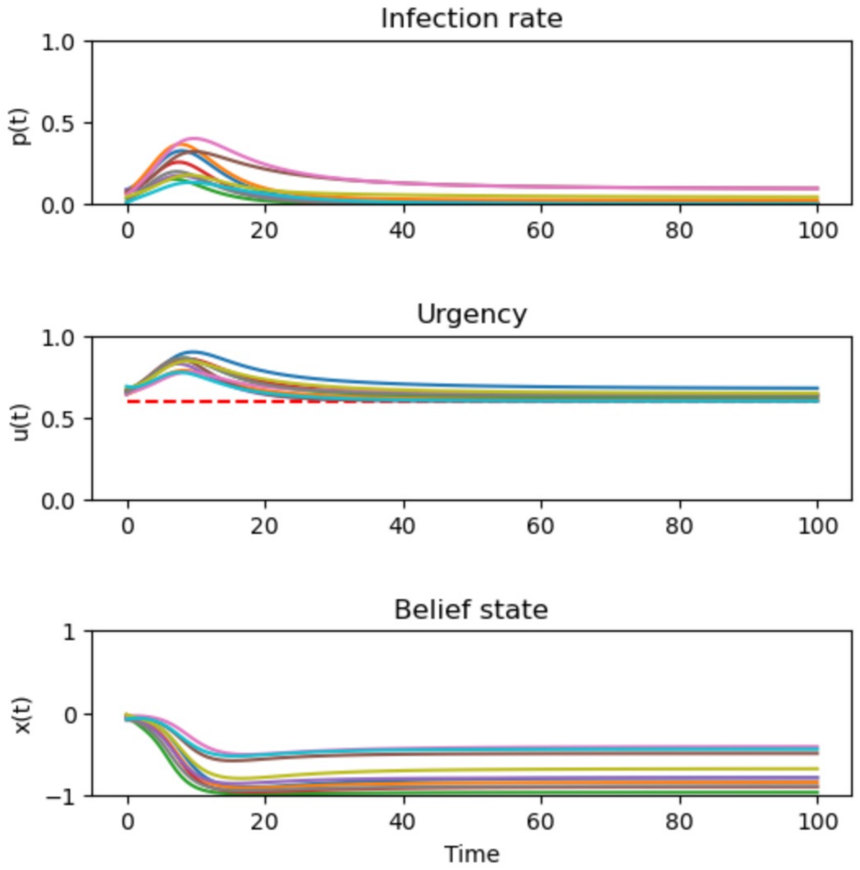
Applications

US Congress--Political polarization



Leonard, Lipsitz, Bizyaeva, Franci, Lelkes, "The nonlinear feedback dynamics of asymmetric political polarization," *PNAS,* 2021

Active risk aversion in SIS epidemics on networks



Bizyaeva, Odorico, Zhou, Levin, Leonard, "Active risk aversion in SIS epidemics on networks," *in prep*

Applications

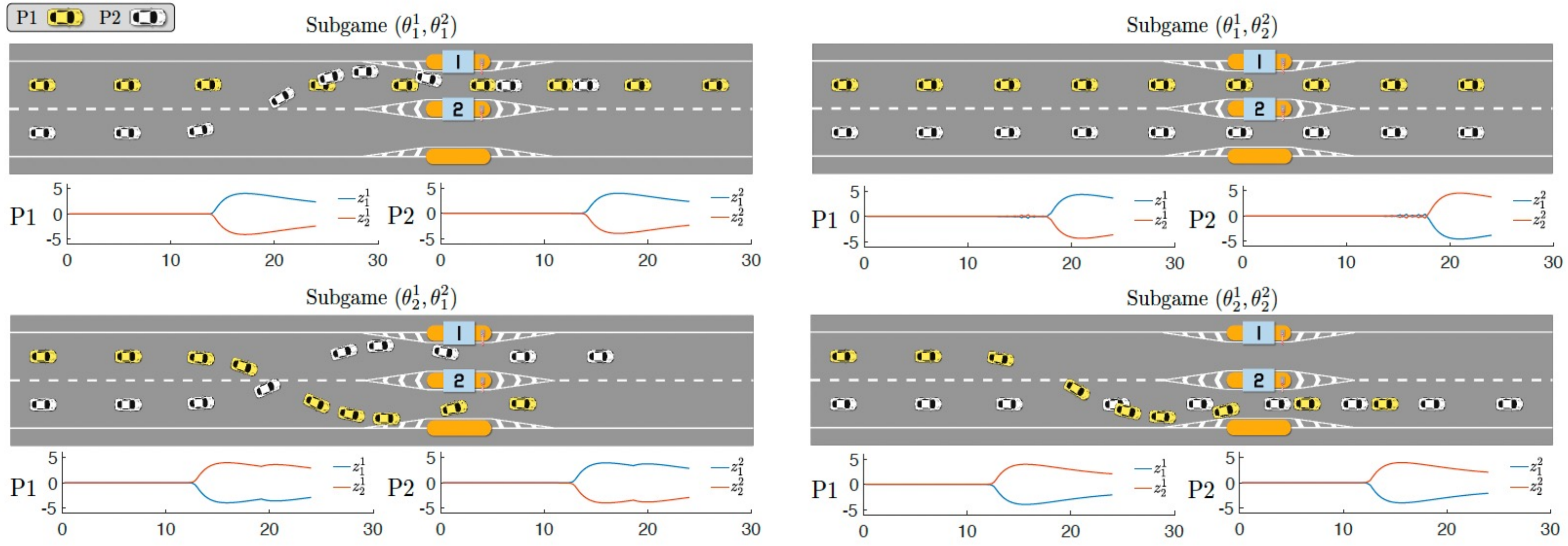
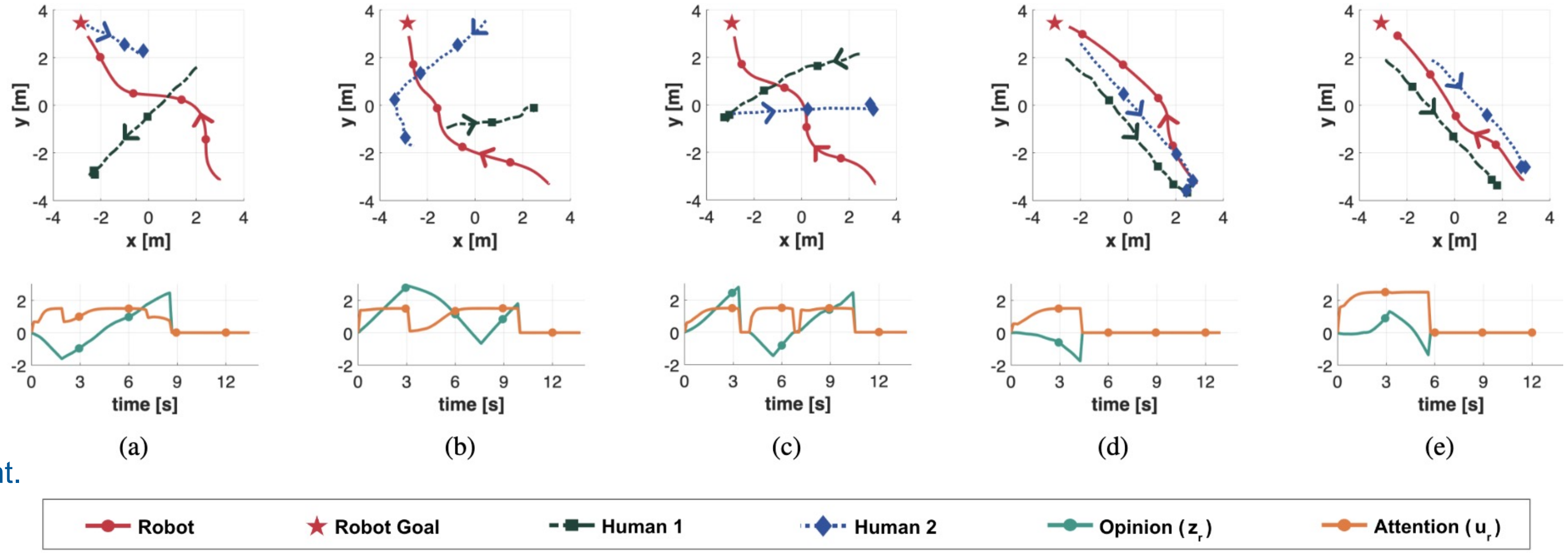


Fig. 2: State and opinion trajectories in four subgames of the Running Example.

Hu, Nakamura, Hsu, Leonard, and Fisac, "Emergent coordination through game-induced nonlinear opinion dynamics", *IEEE CDC,* 2023

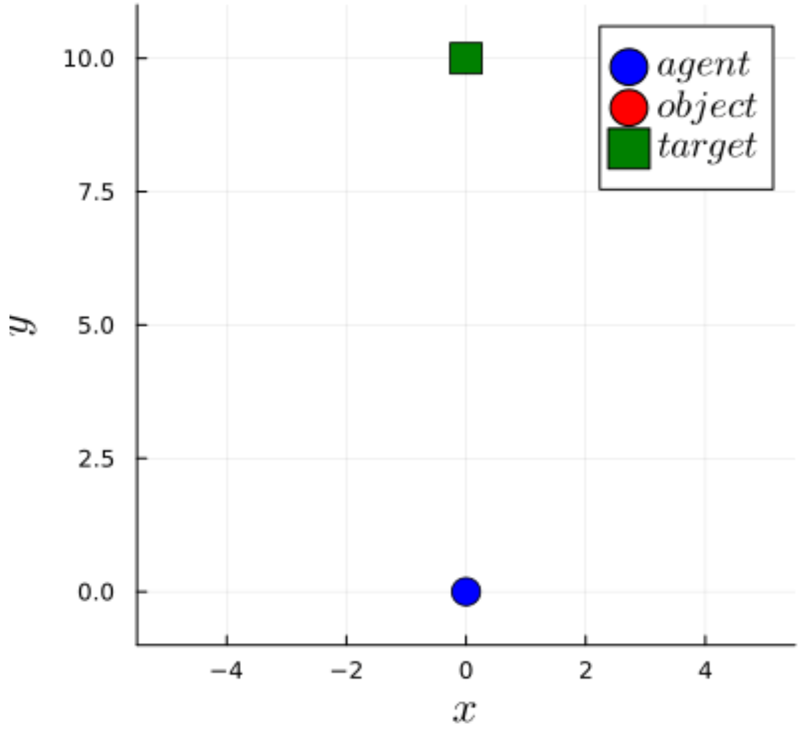


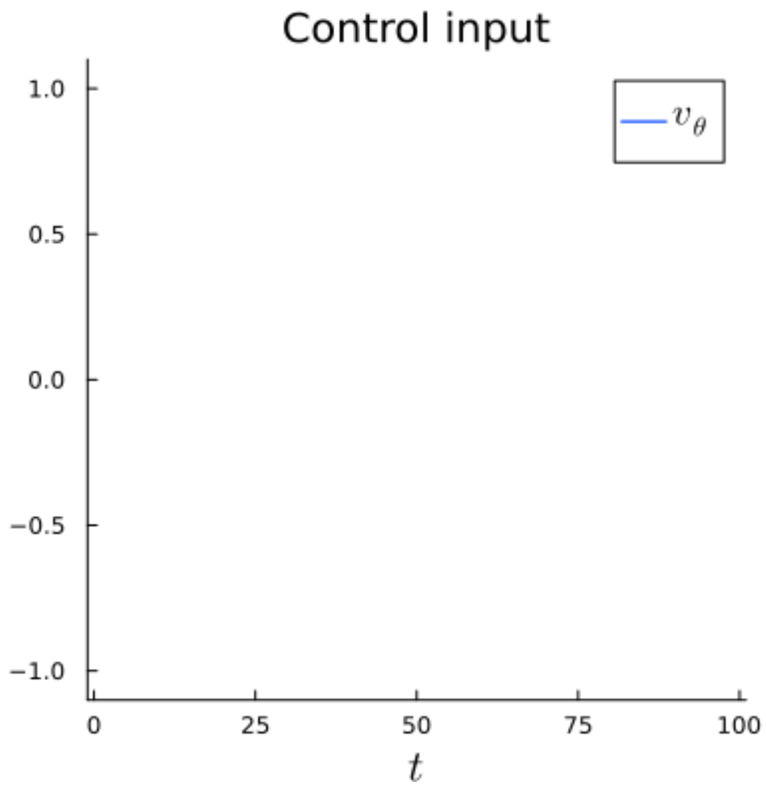
Cathcart, Santos, Park, and Leonard, "Proactive opinion-driven robot navigation around human movers", IEEE/RSJ Int. Conf. on Intelligent Robots and Systems (IROS'23)

Excitable Decision-Making for Additional Flexibility

$$u = u_0 + K_u z^2 - u_s$$

$$\frac{du_s}{dt} = \varepsilon \left(K_s z^2 - u_s \right)$$





Franci et al, “Excitable decision-making”, in prep.

Thank you!

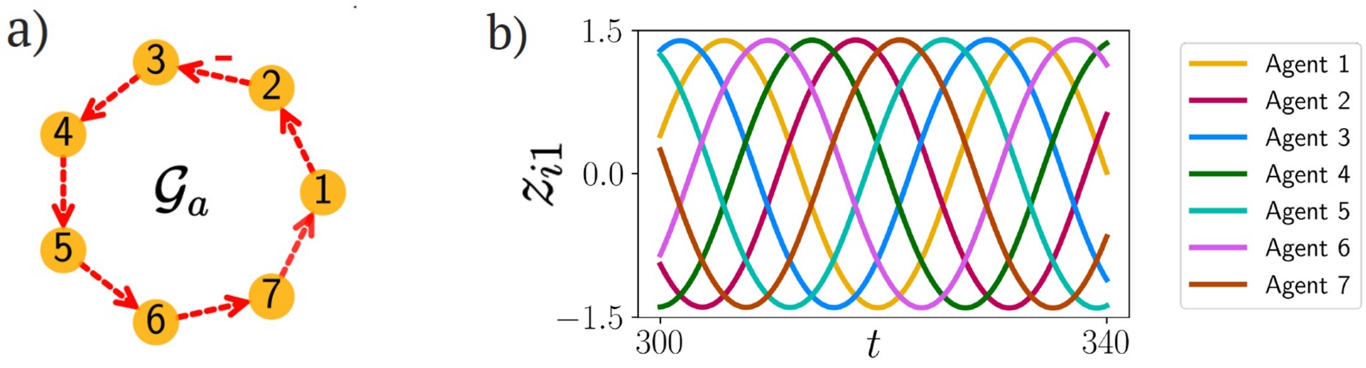
Leonard Lab, May 5, 2022 (Front: Justice Mason, María Santos, Charlotte Cathcart, Isla Xi Han, Back: Christine Ohenzuwa, Giovanna Amorim, N.E. Leonard, Sarah Witzman, Kathryn Wantlin, Anastasia Bizyaeva, Yenan (Daniel) Shen, Mari Kawakatsu, Justin Lidard, Ken Nakamura; missing: Christine Allen-Blanchette, Udari Madhushani)

Tim, Amara, and Lily Leonard

N Agent Opinion Dynamics

When leading eigenvalues of A are a *complex conjugate pair*,
indecision-breaking bifurcation can be a Hopf bifurcation which can lead to sustained oscillations

Two option example:



Bizyaeva, Franci, Leonard, "Sustained oscillations in multi-topic belief dynamics over signed networks", *Proc. ACC*, 2023