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Department of Electronics and Telecommunications

POLITECNICO DI TORINO

Contagion-mitigating control in dynamic financial networks

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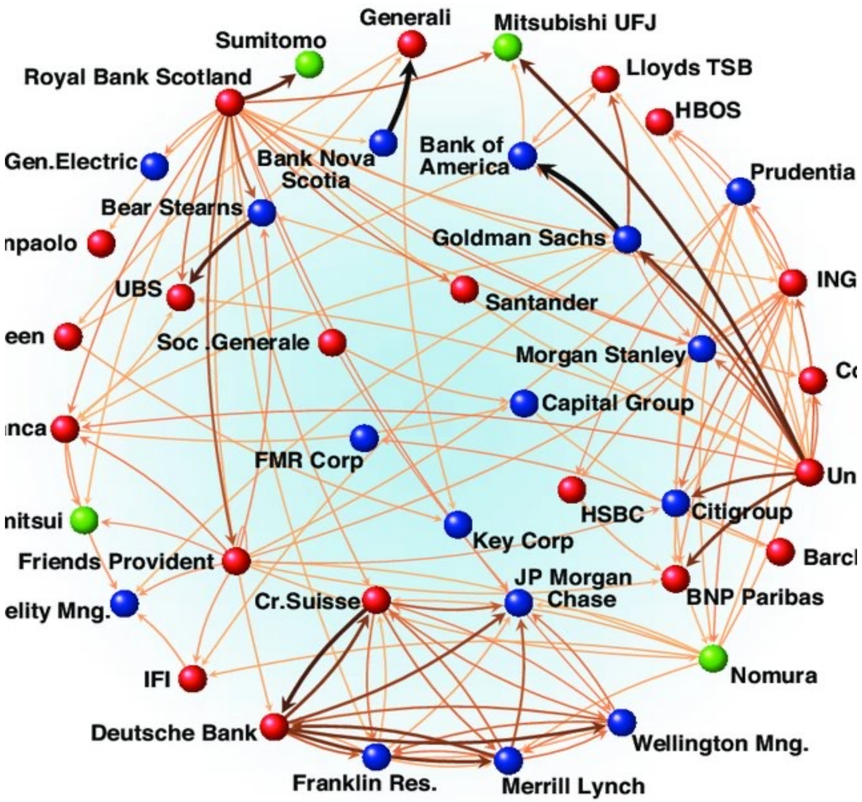
Joint works with G. Calafiore and G. Fracastoro

- Control of Dynamic Financial Networks (The Extended Version), arXiv:2205.08879 (published in L-CSS)
- Clearing Payments in Dynamic Financial Networks, arXiv:2201.12898 (accepted by Automatica)
- Optimal Clearing Payments in a Financial Contagion Model arXiv:2103.10872 (under review)

General motivation. Systemic risk in financial networks

- Highly interconnected structure.
- Complex structure of mutual obligations, shares in common assets etc.
- Many channels of financial "contagion".

- Single fault can threat to the stability of the entire financial system



Schweitzer et al., Economic Networks: The New Challenges, Science, 325(5939)

https://marketbusinessnews.com/financial-glossary/systemic-risk-definition-meaning/

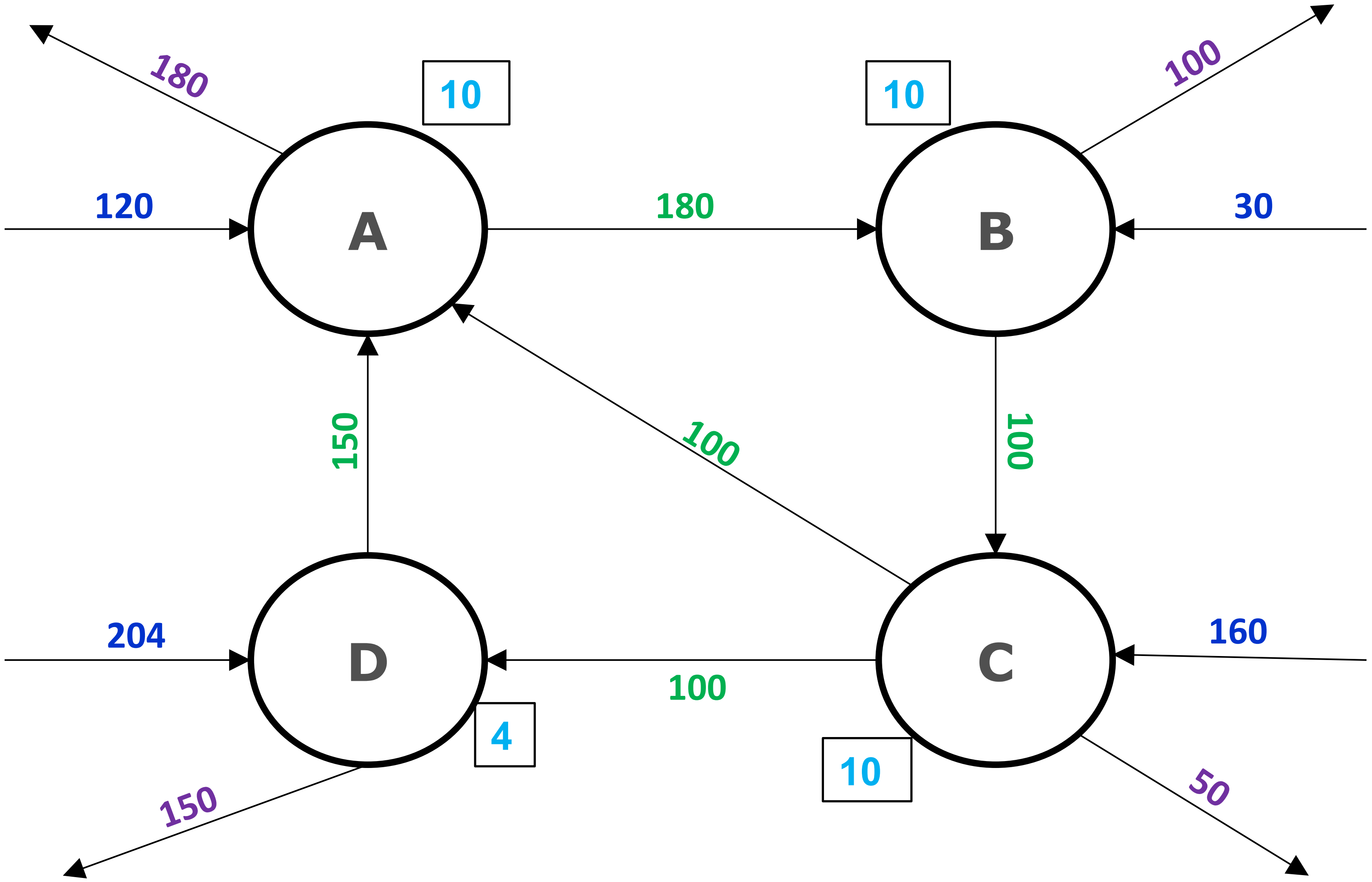
Systemic risk theory:

How stresses, such as bankrupts and failures, in one part of a financial system can spread to its other parts?

Structure of the talks: 3 topics

- **Eisenberg-Noe model: contagion via propagation of liquidity shortage. Clearing payments.**
- **Multi-stage dynamic generalization of the E-N model: clearing as optimal control**
- **Contagion-mitigating interactions**

A simple numerical example: Propagation of liquidity shortages (I)

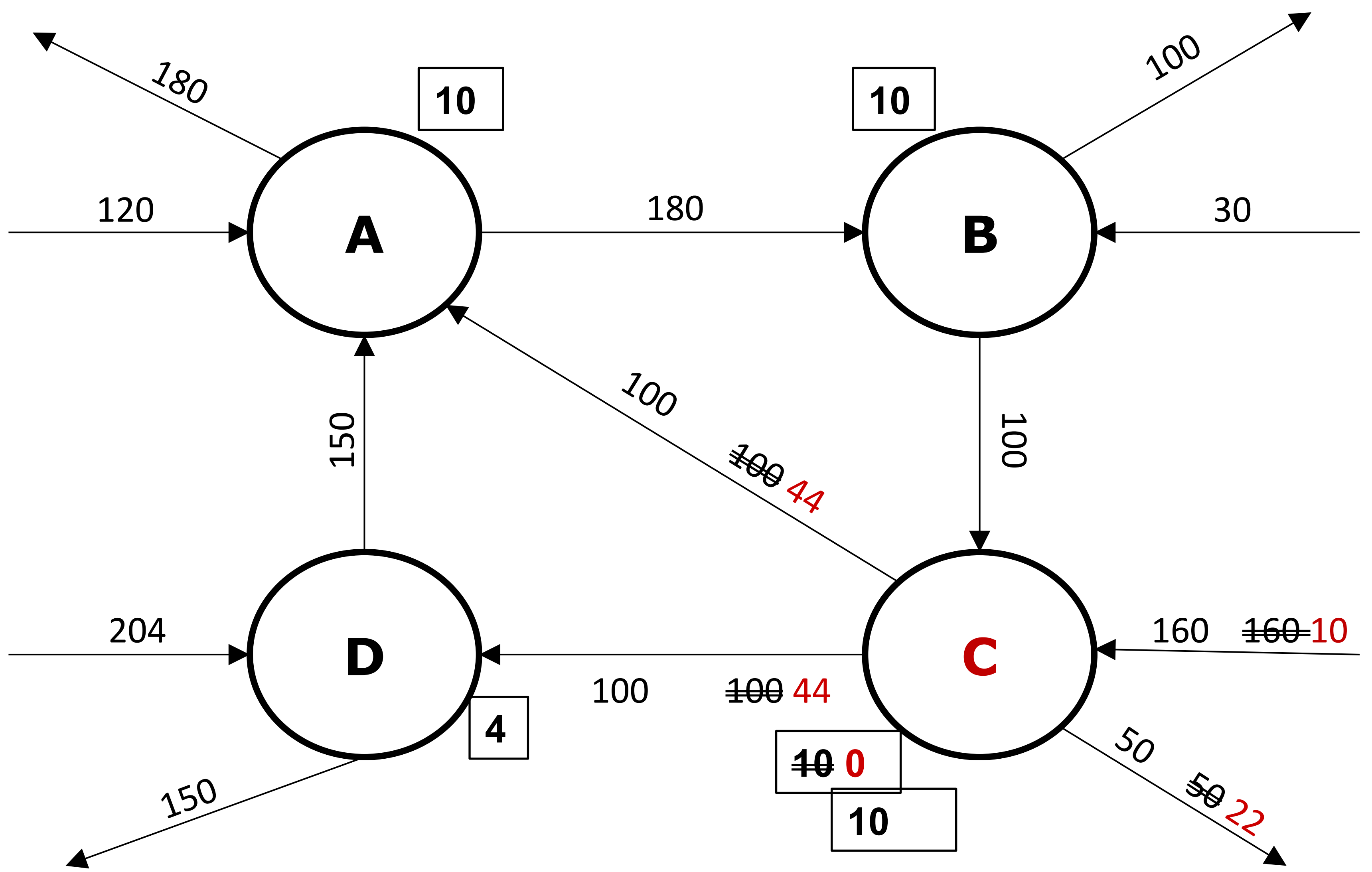


System of banks A,B,C,D

- **Outside assets** (arcs coming from outside**), e.g.** bank A is owed 120 by households and companies
- Arcs = **payment obligations of one bank** to **another bank** (some liquidity provided via loans etc.)
- Obligation to the **external sector** (households, companies etc.)
- The difference between assets and liabilities is a **net worth** (the net worths of banks A,B,C are 10, the net worth of D is 4).
- **Net worth cannot be negative**

Glasserman and Young, Contagion in Financial Networks, Journal of Economic Literature 54(3)

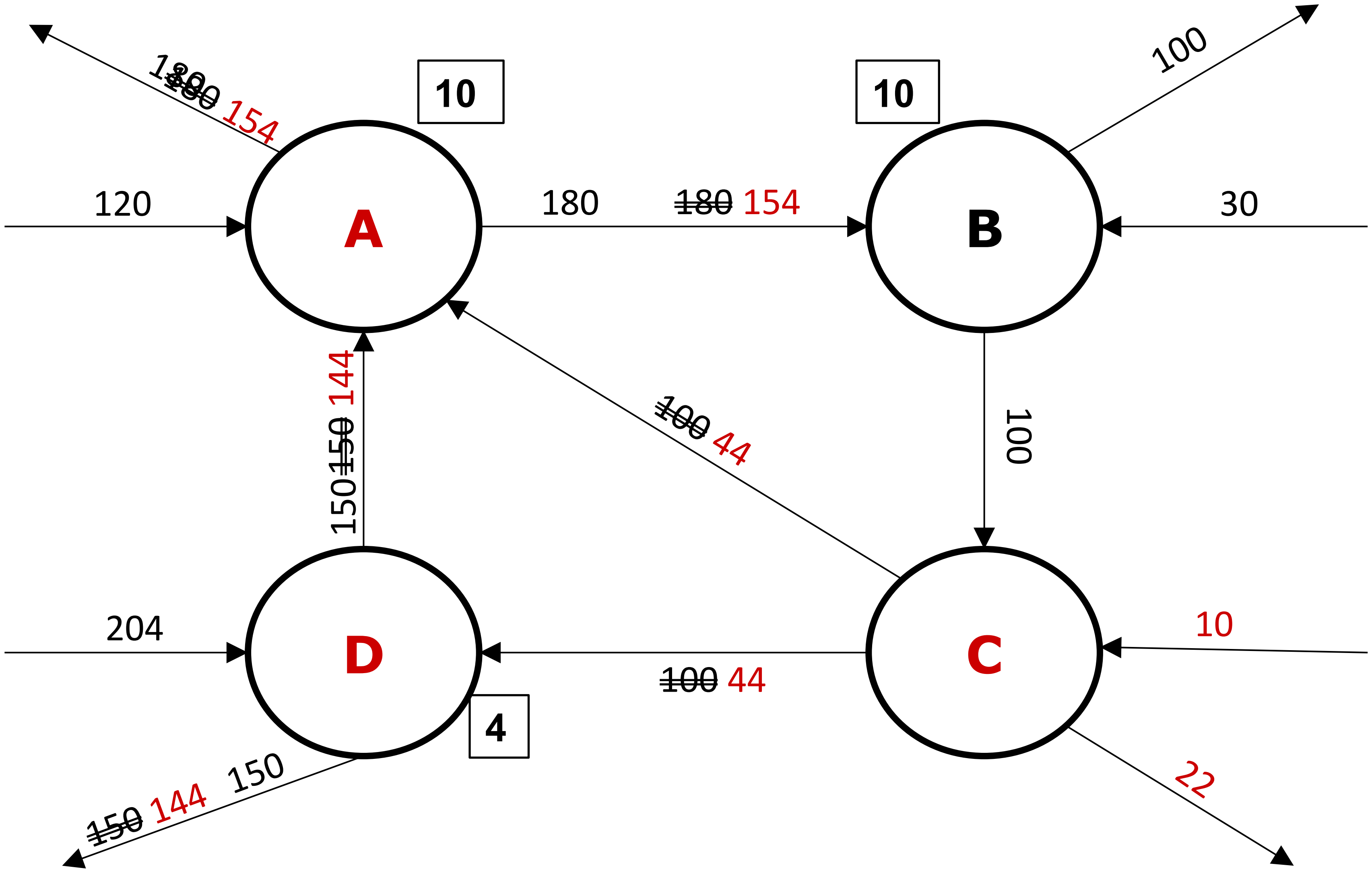
A simple numerical example (II)



- Occasionally, the assets of some bank drop, e. g. households delay or stop their payments. Then the total asset becomes less than the debt to pay.
- Bank has to reduce all payments to the creditors. Usually, **proportionality (pro rata) rule** is applied: C pays to A,D, external sector in the proportion 2:2:1
- Assumption: external payments can be reduced.
- The defaulting bank has to pay the maximal possible amount (**priority rule**), it cannot accumulate its net worth.

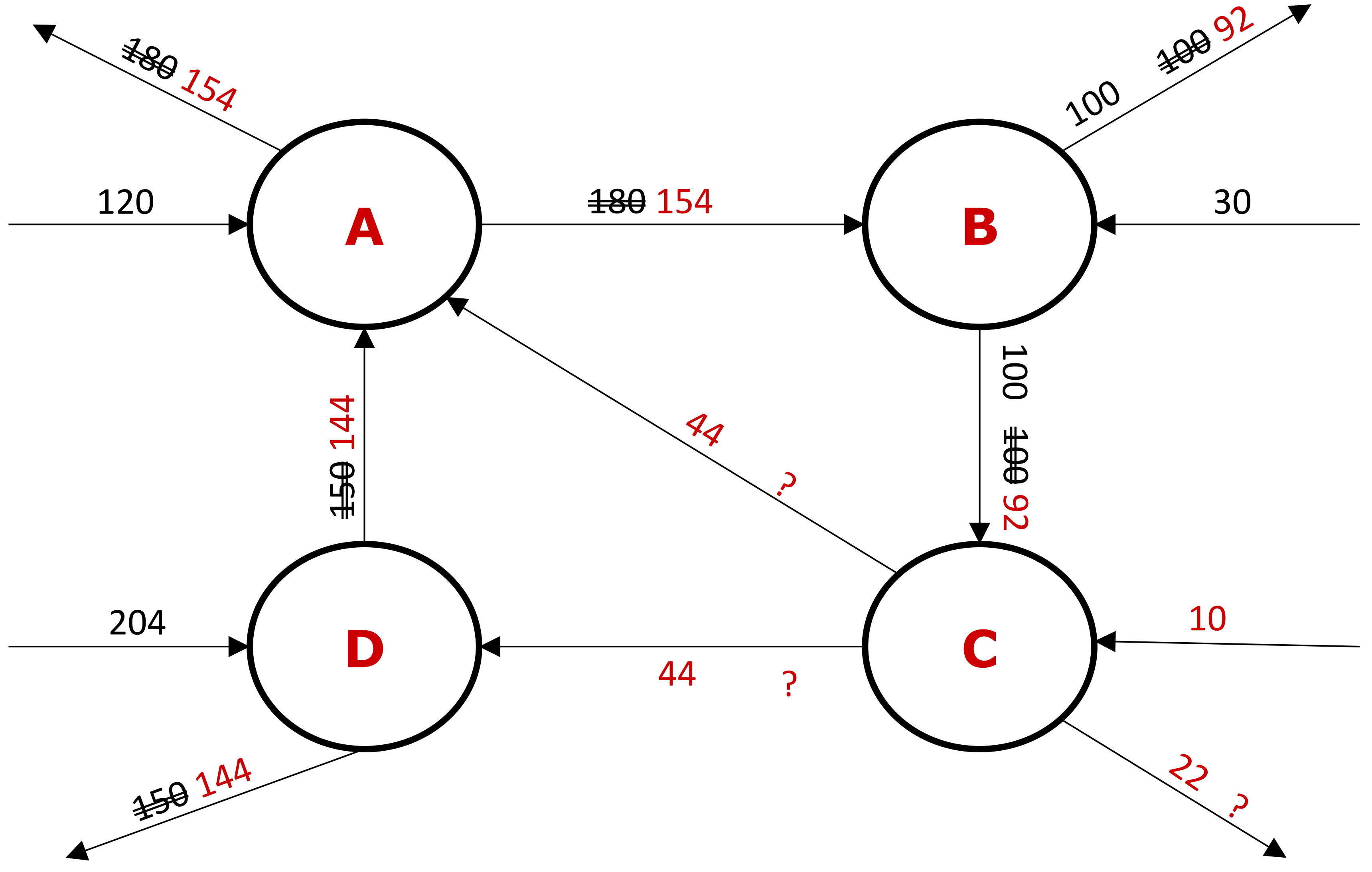
Glasserman and Young, Contagion in Financial Networks, Journal of Economic Literature 54(3)

A simple numerical example: How defaults propagate (III)



- Hence, problems of a single bank cause problems to its creditors whose assets **have decreased.**
- These banks, in turn, have to reduce their payments to their creditors and to the outside sector, which may cause an «avalanche» of defaults

A simple numerical example: How defaults propagate (IV)



- This chain of defaults can be cyclic and return to the «problematic» bank.
- We assume that C has 110 as asset, whereas in reality B has to **decrease** its payment! So C will not get more than 102! We need to recalculate its payments
- What is the **fair** structure of payments in the situation where one or several banks default due to the outside assets' drop? How to minimize the total loss?
- In reality, if you continue the iterative procedure, it will **converge to a fixed point**.

Iterative 'fictitious default' algorithm – converges to an equilibrium

- The procedure of iterative payment reduction (previous slides) gives an idea on how one channel of contagion (liquidity shortage) works. Can be turned into a model of defaults propagation, adding technical details (not the goal of E-N work!)

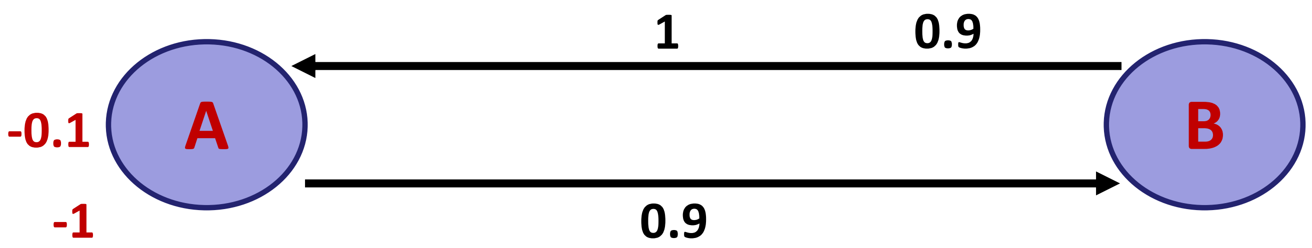
- Can be written as an iteration of a monotone operator on a lattice, a fixed point always exists due to Knaster-Tarski theorem (and, generically, is unique)

- The focus in E-N network is on the **equilibria** states: The **actual** payments after the renegotiations. Needed to evaluate the **consequences and total loss of contagion.**

  Have to be efficiently computable.

- **Clearing in systemic risk = total debt reduction via mutual reimbursements**.

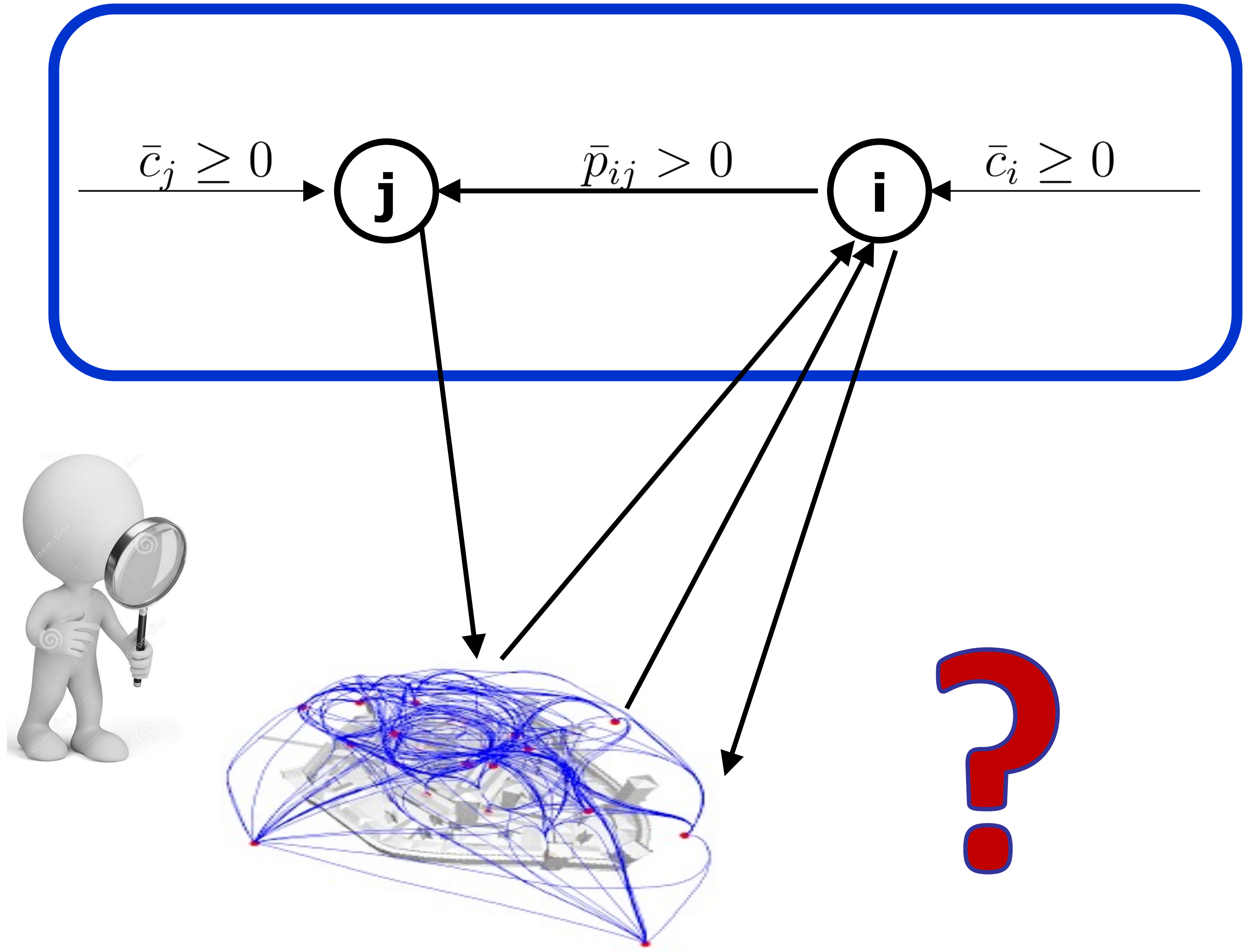
Clearing: a toy example



- Two financial agents A and B
- A lends 1 bln $ to B (directed arc = obligation)
- For some reasons, A needs money and borrows 900 mln $ from B in credit
- If B fails to return its credit due to some problems, then A seeming suffers huge loss…
- But: if A pays its own debt (0.9), then B can repay it back, and the loss of A is 10 times smaller.
- Mutual reimbursements reduce the total loss in the system: the idea of clearing

Y.M. Kabanov et al., Clearing in financial networks//Theory Probab. Appl., 2018, 62(2)

Eisenberg-Noe (2001) model. (i) Obligations



- **Directed graph of obligations**
- **Weights** on arcs = debts to be paid

- **Nominal outside assets** = money to be raised from the non-financial sector

- **Net worth** of node is the difference of
  - the **in-flow**
  - and the **outflow**

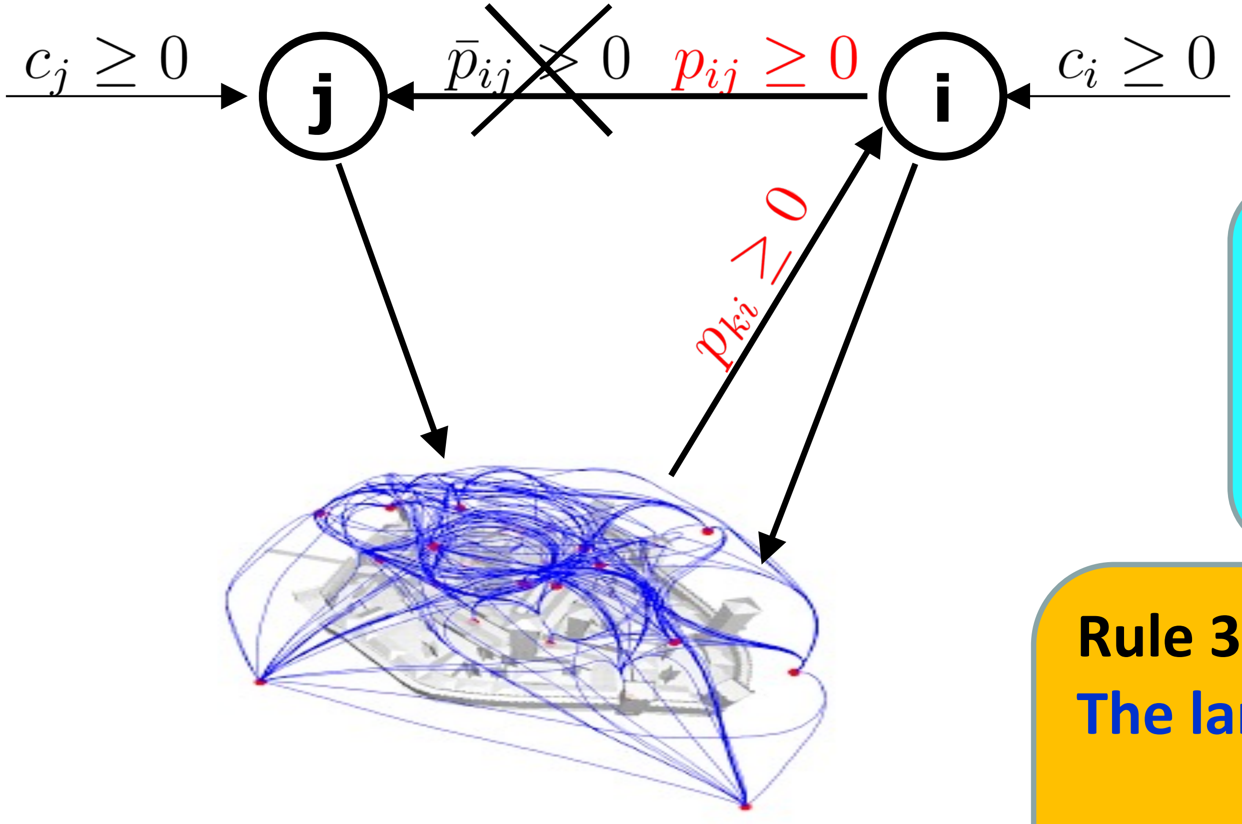
$$\bar{w}_i = \bar{c}_i + \sum_{k \neq i} \bar{p}_{ki} - \sum_{k \neq i} \bar{p}_{ki}$$

  - Normally, the net worth is **non-negative**

- Abnormal case: the external assets are lost or less than their nominal values. Defaults of some banks **propagate through the network.**

- **How much should defaulting banks actually pay to the others?**

Eisenberg-Noe model of a static clearing procedure. (ii) Actual payments



Rule 1: Limited liability. **Payments do not exceed their nominal values, the net value of each bank remains nonnegative.**

$$0 \leq p_{ij} \leq \bar{p}_{ij}, \quad p_i = \sum_{j \neq i} p_{ij} \leq c_i + \sum_{k \neq i} p_{ki}$$

Rule 2: Absolute debt priority.

Bank pays out its full debt $\bar{p}_i$ or its balance.

$$p_i = \min\left(\bar{p}_i, c_i + \sum_{k \neq i} p_{ki}\right). \quad (+)$$

$$\bar{p}_i = \sum_{k \neq i} \bar{p}_{ik}.$$

Rule 3: Proportionality (pro-rata) rule.

The larger debt, the larger actual payment is

$$p_{ij} \sim \bar{p}_{ij} \Longleftrightarrow p_{ij} = a_{ij} p_i, \quad a_{ij} = \frac{\bar{p}_{ij}}{\bar{p}_i}. \quad (*)$$

- **Guarantees each bank a proportion in its debtors' assets.**
- **Reduces the number of unknowns: matrix determined by *p***

$$p_i \overset{(+),(*)}{=} \min(\bar{p}_i, c_i + \sum_{k \neq i} a_{ki} p_k) \; \forall i.$$

A bunch of problems addressed in financial mathematics literature:

- **Does the solution (clearing vector) exist?**
- **Is it unique?**
- **How to find it?**

- **What if we discard the pro-rata rule or replace it by an alternative division rule? Will this mitigate the loss caused by the defaults?**

Existence of the clearing vector: nonlinear equations are solvable

$$p_i = \min(\bar{p}_i, c_i + \sum_{k\neq i} a_{ki} p_k)\ \forall i \iff p = T_c(p)$$

***p* is called a clearing vector.**

- **The set of clearing vectors is always non-empty**
- The maximal (**or dominant**) clearing vector *p** exists such that dominates any other clearing vector *p,* that is, ***p* ≤ *p**.**
- Furthermore, *p** is the maximal (with respect to ≤) element of the polytope

$$\mathcal{D} = \{p \in \mathbb{R}^N : 0 \leq p \leq \bar{p},\ c + A^\top p \geq p\} \neq \emptyset.$$

- In particular, *p** delivers optimum in the LP problem

$$L(p) = \sum_{i,j} (\bar{p}_{ij} - p_{ij}) = \sum_i (\bar{p}_i - p_i) \to \min \quad \text{subject to}\, p \in \mathcal{D}.$$

The **"loss function"** *L(p)* can be replaced by any strict decreasing function.

- Alternative way: the fixed-point iteration ("fictitious default algorithm")

$$p(n+1) = T_c(p(n)),\ \ p(0) = \bar{p}$$

G. Calafiore et al. "Optimal clearing payments in a financial contagion model," **arXiv:2103.10872**.

Uniqueness of the clearing vector: sufficient conditions

$$p_i = \min(\bar{p}_i, c_i + \sum_{k \neq i} a_{ki} p_k) \, \forall i \overset{?}{\Longrightarrow} p = p^*.$$

- The clearing vector may be non-unique. Such situations are, however, **non-generic**.

- The clearing vector is unique if and only if every non-trivial (>1 node) strongly connected sink component either contains a node with $c_i > 0$

  or

  is reachable from one of such nodes. **Some graph theory and matrix analysis are needed.**

- Otherwise, it is possible to describe the set of all clearing vectors

G. Calafiore et al. "Optimal clearing payments in a financial contagion model," online as **arXiv:2103.10872**.

- More general cases (negative vector *c* or replacement by pro rata by other division rules): Massai, Como, Fagnani (2022); Herings and P. Csoka (2021)

The price of pro-rata rule.

- **Pro-rata rule is often considered as natural for a number of reasons and is included in the bankruptcy law in many countries. Signing a contract, we want to guarantee some portion of the debtor's assets in case of its fault.**

- The pro-rata rule visible reduces the set of clearing matrices.
- It can be expected that the overall loss can be substantially decreased by removing it.

- We can find the clearing matrix delivering minimum to the overall loss (also reduces to LP, see details in the proceedings). Optimal matrix is generally non-unique.

$$L(p) = \sum_{i,j} (\bar{p}_{ij} - p_{ij}) = \sum_{i} (\bar{p}_i - p_i) \to \min .$$

- We compare the optimal losses with pro-rata rule and without pro-rata rule on synthetic random networks. The model is borrowed from: *E. Nier, J. Yang, T. Yorulmazer, and A. Alentorn, "Network models and financial stability," J. Econ. Dynamics and Control, vol. 31, no. 6, pp. 2033–2060, 2007*

The price of pro-rata rule: numerical tests

- The standard Erdős–Rényi random graph G(n,p) is considered, *n=50*, average node degree varies from 0 to *np=35*. One fictitious sink node is added.
- Nominal liability of each arc chosen at random from [0,100] (uniform distribution).
- Vector *c* designed to provide positive yet small net worths (see details in the proceedings).

- Shocks are applied: outside asset of **one randomly chosen** bank is nullified.

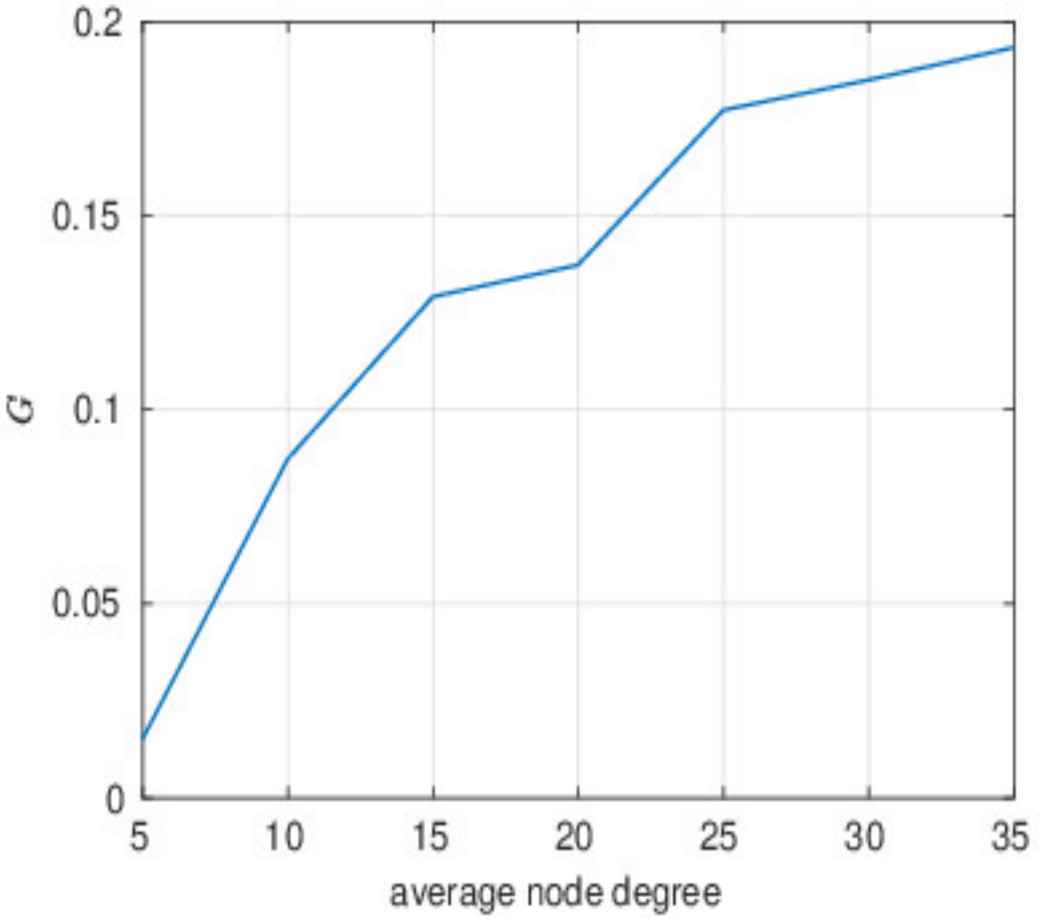
- The following ratio may be considered as the "gain" of the pro-rata rule discarding:

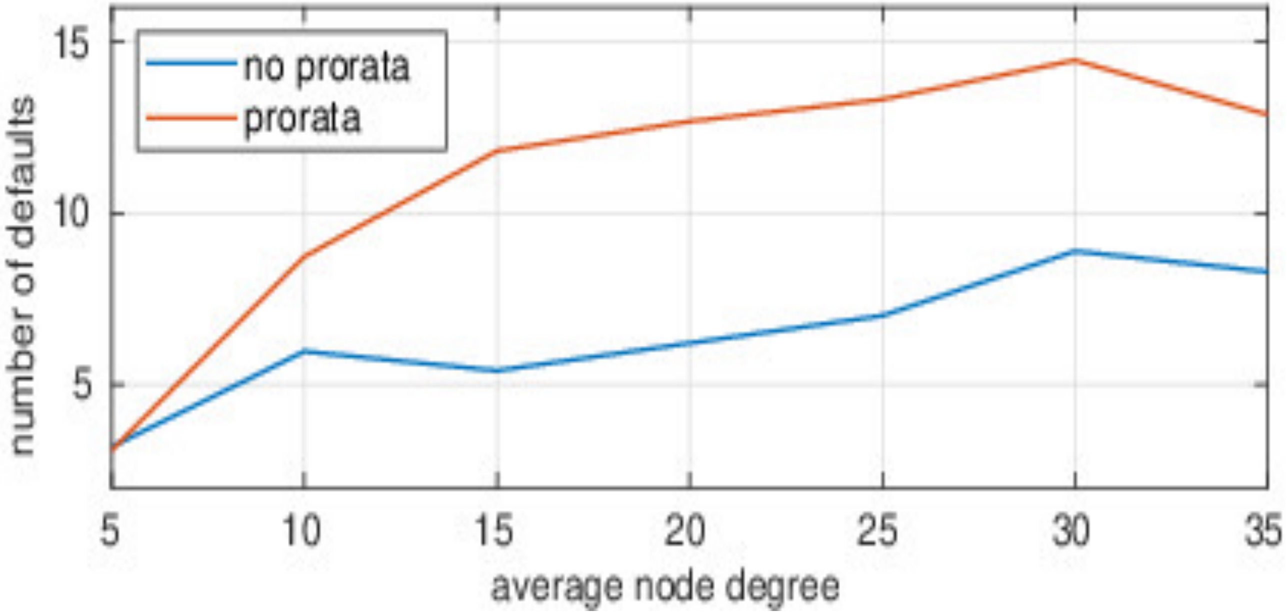
$$G = \frac{L_{min,pro-rata} - L_{min}}{L_{min,pro-rata}} = 1 - \frac{L_{min}}{L_{min,pro-rata}}$$

- Another measure is the number of defaulting banks under the optimal choice of payments. Bank defaults if its actual payment is less than the nominal one: $p_i < \bar{p}_i$

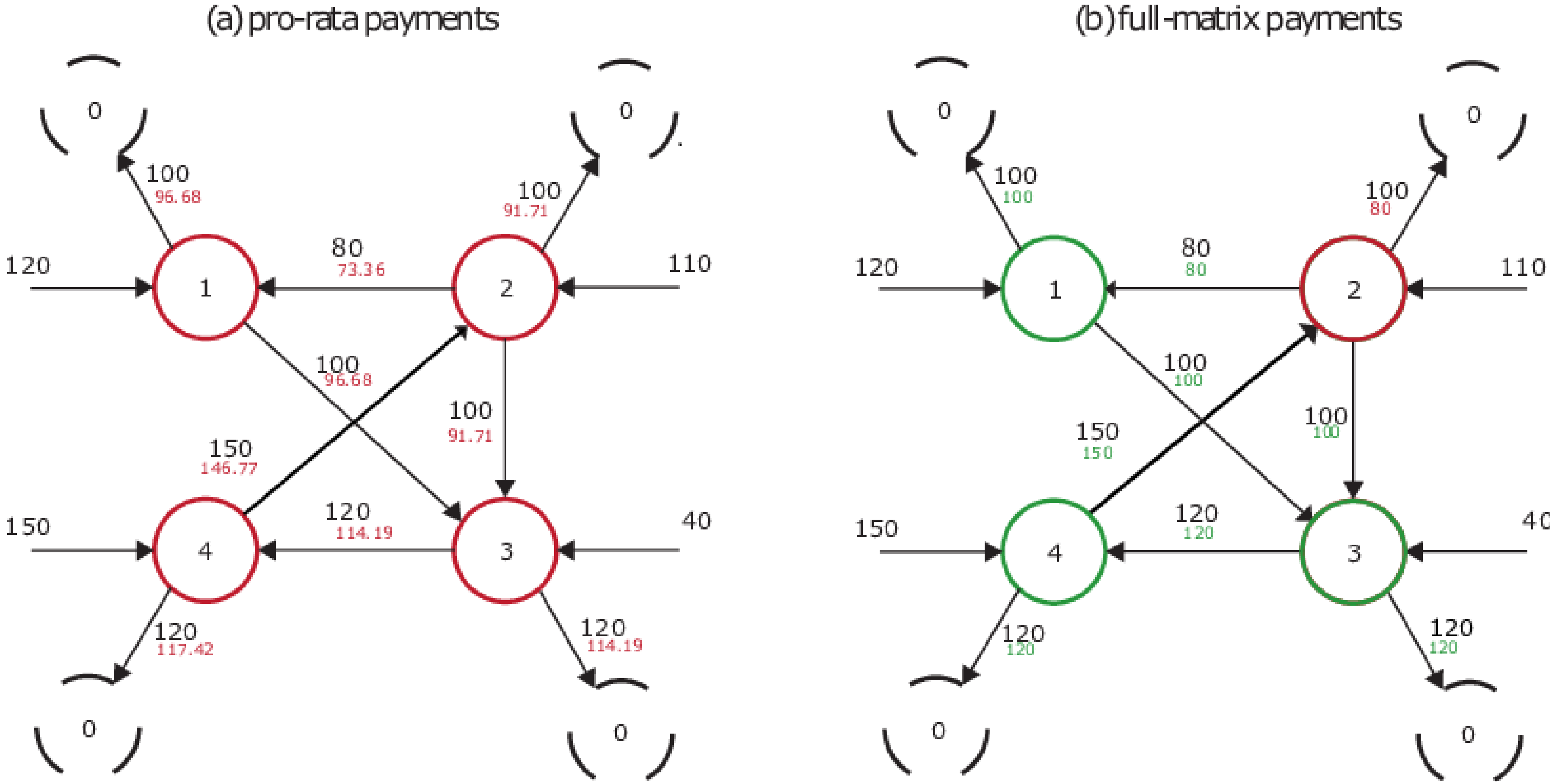
The price of pro-rata rule: fair locally, non-optimal globally

- Visible decrease of overall loss as the graph becomes denser (the average degree *np* grows). The number of defaults without pro-rata rule is also less than with pro-rata rule.





Pro-rata vs. free payment matrix (isolation of a problematic node)



All banks in default, total unpaid amount is 47.28

1 bank in default, total unpaid amount is 20

P. Glasserman and H. P. Young, "Contagion in financial networks," Journal of Economic Literature, vol.54, no.3, pp.779–831, 2016

0 – fictitious node, corresponds to the payments to non-financial sector

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- **Eisenberg-Noe model: contagion via propagation of liquidity shortage. Clearing payments.**
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Principal limitation of the E-N model

The EN model describes static clearing process

- all outside assets are available at once;
- all liabilities are claimed and due simultaneously;
- all clearing payments are computed simultaneously;

Next step: non-static (multi-step) clearing models:

- **financial operations are allowed for a given number of time periods after the initial shocks;**
- **some nodes may actually recover and eventually manage to fulfill their obligations if the liquidity inflow continues;**
- **we do not freeze operations in case of instantaneous default, but allow for a grace period and carry over the residual liabilities for the next time slot**

Model 1: Optimal multi-period clearing procedure

operations over ***T*** consecutive periods

The optimal choice of *P(t)*: minimal total loss

$$L = \sum_{t=0}^{T-1} \sum_{i,j} (\bar{p}_{ij}(t) - p_{ij}(t)) \to \min$$

at each period ***t=0,1,…, T-1***, bank ***i*** is characterized by:

- outside asset: external input $c_i(t) \geq 0$
- nominal liabilities to other banks: dynamic variable $\bar{p}_{ij}(t) \geq 0$
- net worth in the **previous** periods: dynamic variable $w_i(t)$

Initial values

$$\bar{P}(0) = \bar{P}$$
$$w(0) = \mathbf{0}$$

The **actual payments are also distributed over *T* periods** $p_{ij}(t)$

- **Limited liability rule (pay no more than liability, the balance remains non-negative**

$$0 \leq p_{ij}(t) \leq \bar{p}_{ij}(t) \qquad w_i(t+1) = w_i(t) + c_i(t) + \sum_{k \neq i} p_{ki}(t) - \sum_{k \neq i} p_{ik}(t) \geq 0$$

- We do not impose absolute debt priority (non-convex!)
- The residual liabilities are transferred to future periods, and the interest rate can apply

$$\bar{p}_{ij}(t+1) = \alpha(\bar{p}_{ij}(t) - p_{ij}(t))$$

- **Pro-rata rule with the proportions are determined at the initial step**

$$p_{ij}(t) = a_{ij} p_i(t) \qquad a_{ij} = \bar{p}_{ij} / \bar{p}_i \qquad p_i(t) = \sum_{j \neq i} p_{ij}(t)$$

Optimal multi-period clearing procedure: equivalent LP

$$L = \sum_{t=0}^{T-1} \sum_{i,j} (\bar{p}_{ij}(t) - p_{ij}(t)) = a_0 \mathbf{1}^\top \bar{p} - \sum_{t=0}^{T-1} a_t \mathbf{1}^\top p(t) \to \min_{p(0),\dots,p(T-1)}$$

$$a_t \doteq \sum_{j=0}^{T-t-1} \alpha^j = \begin{cases} \frac{\alpha^{T-t}-1}{\alpha-1}, & \text{if } \alpha > 1 \\ T-t, & \text{if } \alpha = 1 \end{cases} \qquad (a_0 > a_1 > \ldots > a_{T-1})$$

subject to

$$p(t) \geq 0,$$

$$\sum_{k=0}^{t} \alpha^{t-k} p(k) \leq \alpha^t \bar{p}$$

$$\sum_{k=0}^{t} c(k) + \sum_{k=0}^{t} \left(A^\top p(k) - p(k)\right) \geq 0$$

$$\forall t = 0, 1, \ldots, T-1.$$

Limited liability (+dynamics)

$$0 \leq p_{ij}(t) \leq \bar{p}_{ij}(t)$$

$$w_i(t+1) \geq 0 \, \forall i$$

Optimal solution – the counterpart of the maximal clearing vector

- Optimal solution exists and is unique for each sequence of outside assets ***c(0),…,c(T-1):*** (a non-trivial property: solution to LP may be non-unique!)
- **Absolute debt priority is implied by the optimality:**

$$p_i^*(t) = \min\left(\bar{p}_i(t), w_i(t) + c_i(t) + \sum\nolimits_{k\neq i} p_{ki}^*(t)\right), \quad p_{ki}^*(t) = a_{ki} p_k^*(t).$$

- Causality: in fact, the optimal value ***p*(t)*** depends on ***c(0),…,c(t)***
- **Greedy strategy is optimal**: in fact, ***p*(t)*** minimizes the loss in the static E.-N. problem

$$L_t(p) = \sum\nolimits_i (\bar{p}_i(t) - p_i) \to \min_p$$

$$\text{subject to}$$

$$0 \le p \le \bar{p}(t),\ w(t) + c(t) + A^\top p \ge p.$$

- Instead of solving LP with ***Tn*** scalar variables and ***3Tn*** constraints**,** one can solve a sequence of ***T*** LP with ***n*** variables and ***3n*** constraints:

$$[\bar{p} = \bar{p}(0), w(0) = 0] \xrightarrow{c(0)} p^*(0) \longrightarrow [\bar{p}(1), w(1)] \xrightarrow{c(1)} p^*(1) \longrightarrow [\bar{p}(2), w(2)] \xrightarrow{c(2)} \ldots$$

Model 2: Same as Model 1, but without pro-rata rule

$$L = \sum_{t=0}^{T-1} \sum_{i,j} (\bar{p}_{ij}(t) - p_{ij}(t)) = a_0 \mathbf{1}^\top \bar{P} \mathbf{1} - \sum_{t=0}^{T-1} a_t \mathbf{1}^\top P(t) \mathbf{1} \to \min_{P(0),\ldots,P(T-1)}$$

subject to

$$P(t) \geq 0,$$

$$\sum_{k=0}^{t} \alpha^{t-k} P(k) \leq \alpha^t \bar{P}$$

Limited liability (rewritten)

$$\sum_{k=0}^{t} c(k) + \sum_{k=0}^{t} \left(P(k)^\top \mathbf{1} - P(k) \mathbf{1} \right) \geq 0$$

$w_i(t+1) \geq 0 \, \forall i$

$$\forall t = 0, 1, \ldots, T-1.$$

Properties of the optimal solution without the pro-rata rule

- Optimal solution exists but is non-unique even for T=1
- No causality: in fact, the optimal matrices ***P*(t)*** depend on all ***c(0),…,c(T-1)***
- **Absolute debt priority:** $p_i^*(t) = \min\left(\bar{p}_i(t), w_i(t) + c_i(t) + \sum_{k \neq i} p_{ki}(t)\right)$
- **Greedy strategy is generally sub-optimal,** the LP **cannot be solved** as a sequence of T smaller problems
- The key advantage: the total loss and number of defaulting banks reduces

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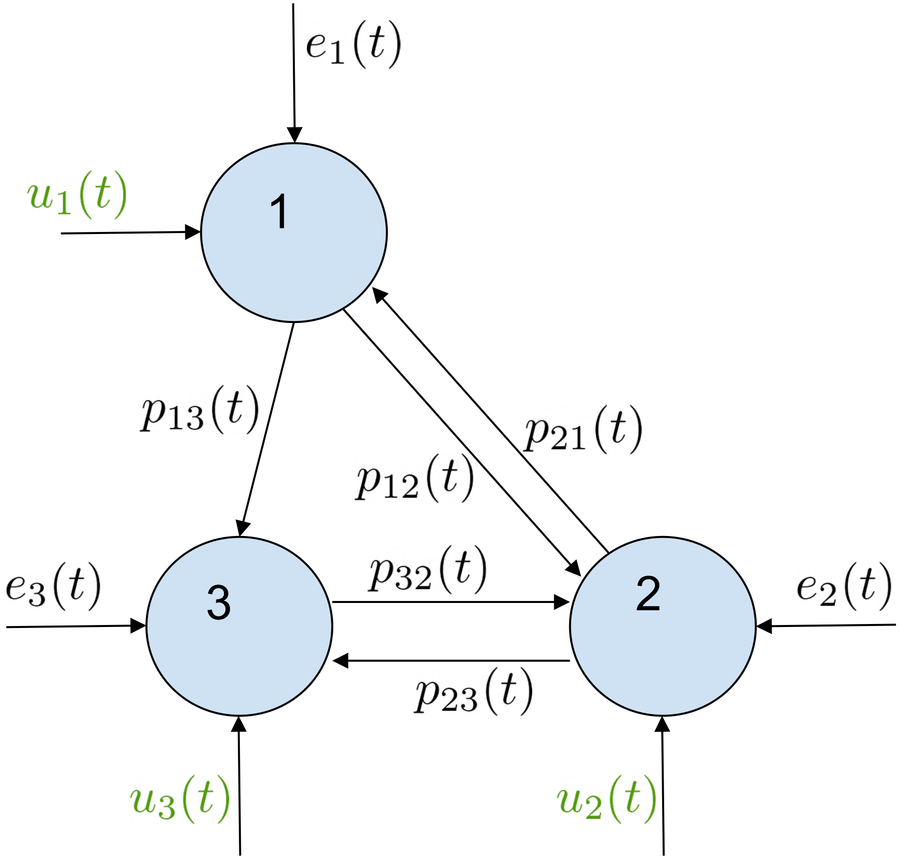
Additional possibility: mitigation of the shock effect by liquidity injections

- Some authorities (central banks etc.) can **control** the financial network by optimal injections of cash at nodes

$$[u] = (u(0), \ldots, u(T-1))$$

$$c(t) = e(t) + u(t)$$



Cost function: same as before or even slightly more general

$$J([p],[u]) = (1-\eta)L([p]) + \eta\mathbf{1}^\top \bar{p}(T) + \gamma B(T-1)$$

- We penalize the total loss (same as before)
- + terminal cost (zero if no bank is at default)
- + the total **budget** used to help the banks

$$B(s) = \sum_{t=0}^{s}\sum_{i=1}^{n} u_i(t)$$

Constraints: same as before + budget

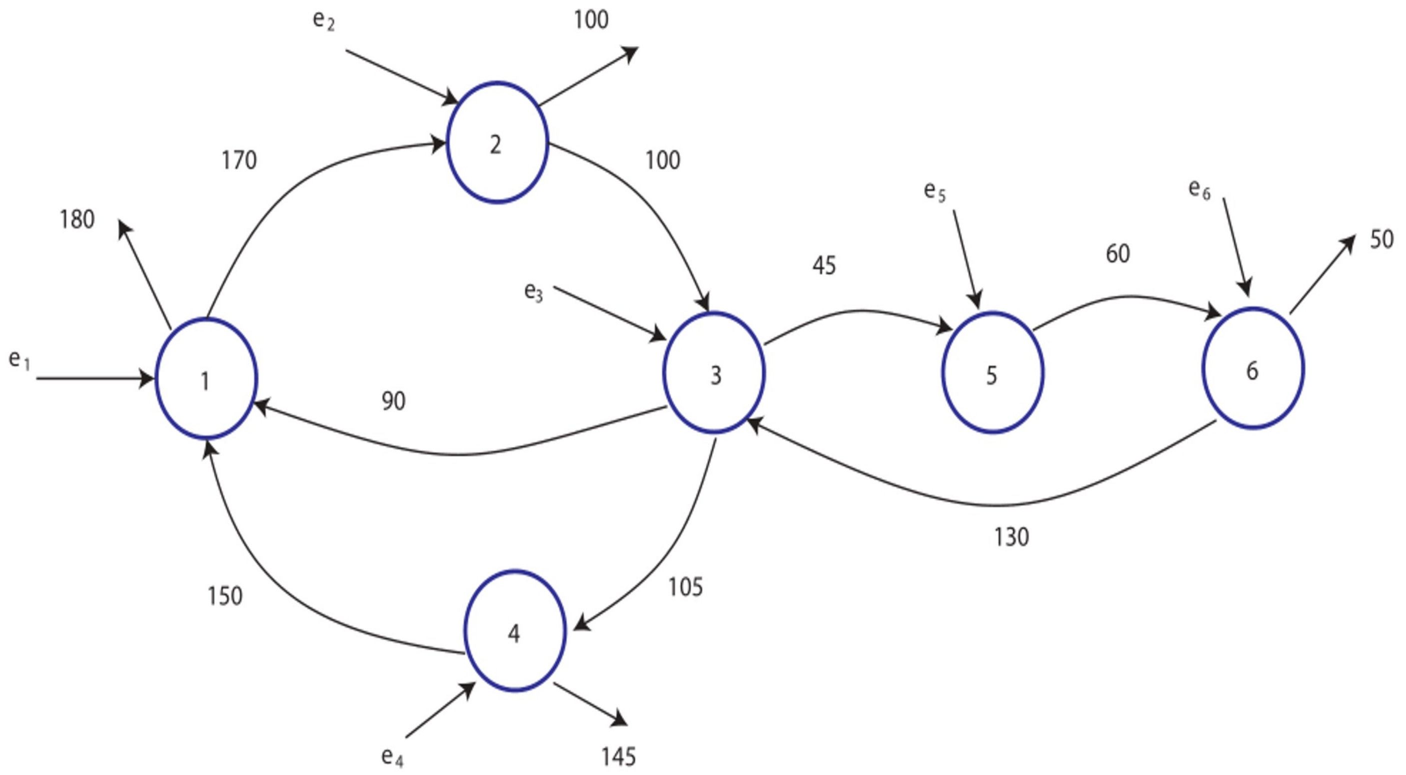
We can define the optimal control problem as an **LP problem**, imposing

- limited liability (rewritten):

$$p(t) \geq 0,$$
$$\sum_{k=0}^{t} \alpha^{t-k} p(k) \leq \alpha^t \bar{p}$$
$$\sum_{k=0}^{t} c(k) + \sum_{k=0}^{t} \left(A^\top p(k) - p(k)\right) \geq 0$$
$$\forall t = 0, 1, \ldots, T-1.$$

- limited budget for controlling the network: $B(t) \leq F(t)$

Numerical example: injected liquidity vs. potential loss



$$e(0) = (105, 25, 10, 190, 10, 120, 0)$$

- Without interventions **all nodes default,** total loss is 49.92.
- We consider the control problem over *T*=3 periods with *F(0)*=15, *F(1)*=30, *F(2)*=50, the optimal control is

$$u(0) = \begin{bmatrix} 2.19 \\ 5.25 \\ 0 \\ 0 \\ 5.20 \\ 2.36 \\ 0 \end{bmatrix}, u(1) = \begin{bmatrix} 2.84 \\ 0 \\ 0 \\ 1.9 \\ 0 \\ 0 \\ 0 \end{bmatrix}, u(2) = 0$$

- **No bank is at default** at *T*=3
  - total injected liquidity is 19.74.

Properties of optimal solutions

$$J([p],[u]) = (1-\eta)L([p]) + \eta\mathbf{1}^\top\bar{p}(T) + \gamma B(T-1)$$
$$\eta \in [0,1), \gamma > 0$$

- **The debt priority rule is respected:**

$$p_i^*(t) = \min\left(\bar{p}_i(t), w_i(t) + e_i(t) + u_i^*(t) + \sum\nolimits_{k\neq i} p_{ki}^*(t)\right), \quad p_{ki}^*(t) = a_{ki}p_k^*(t).$$

- **The bank utilizes liquidity immediately by paying out all its balance**

$$u_i^*(t) > 0 \implies w_i(t+1) = 0,$$

- **Additional liquidity is provided as early as possible to each bank:**

$$u_i^*(t_0) = 0, B^*(t_0) < F(t_0) \implies u_i^*(t) = 0\,\forall t \geq t_*.$$

Conclusion

- We propose a novel dynamic model of clearing in financial networks.
- The model departs from the classical Eisenberg-Noe model and inherits some its properties, e.g., uniqueness of the optimal solution under the pro-rata rule.
- Relaxing the pro-rata constraint, one can substantially decrease the number of defaulting banks and the total loss, however, the problem becomes more complicated and cannot be solved stepwise.
- We consider optimal control interventions aimed at mitigating the damage of an initial financial shock
- Further extensions: robust control in the face of uncertainties in outside assets and nominal debts. Calafiore et al., "Control of Dynamic Financial Networks," L-CSS, vol. 6, 2022
- Future works: more realistic models (common illiquid assets etc.), MPC-like control


Thank you!