Relative timing and localization of inaccessible swarms

Raj Thilak Rajan

Jointly with Alle-Jan van der Veen, Geert Leus, Anurodh Mishra

Signal Processing Systems, Faculty of EEMCS

Delft University of technology, The Netherlands

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Faculty of EEMCS, Delft university of Technology

Signal Processing Systems, Faculty of EEMCS

Professors

- Alle-Jan van der Veen (Chair)
- Geert Leus – Graph signal processing
- Natasja de Groot – Cardiology (at EMC)

Associate Professors

- Gerard Janssen – Physical layer communications
- Raj Thilak Rajan – Dist. autonomous systems
- Richard Hendriks – Audio/Bio signal processing
- Justin Dauwels – Machine learning

Assistant Professors

- Geethu Joseph – Sparse signal processing

Guest Professors

- Richard Heusdens – (at NLDA)
- Bori Hunyadi – (at UMast.)
- Hadi Jamali Rad– (at NXP)

Researchers

- Postdocs: 5, PhDs: 30

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Master Coordinator for Signal Processing Track, MS-EE
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7 PhD
6 MS

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Autonomous Drone Navigation

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Reliable swarm navigation of drones

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Robust Sensor Fusion for Automotives

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TUD: CRANES (`21-)
Cooperative relative navigation

NWO-NLDA: SPIN (`26-)
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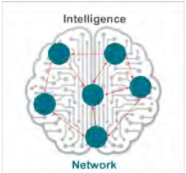

Networked Cyber-Physical Systems (NCPS)
Multi-agent systems (MAS)
Swarms
...

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Overview

Challenges

Automotives

Drones

Satellites

Rovers

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  - Relative positioning/localization of a cooperative mobile network

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Measurement techniques for Localization

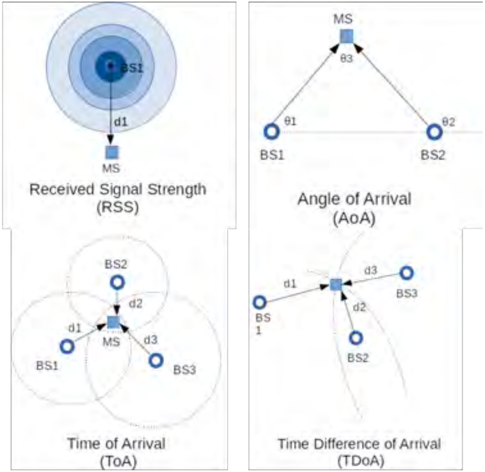

Figure: Conventional paradigms to localize Mobile Station (MS) with Base Stations (BS) or anchors

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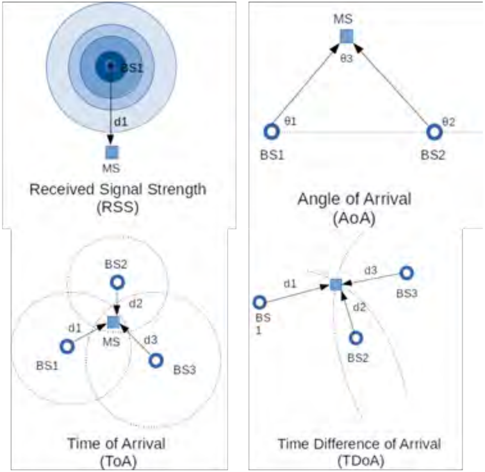

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Relative cooperative timing and localization

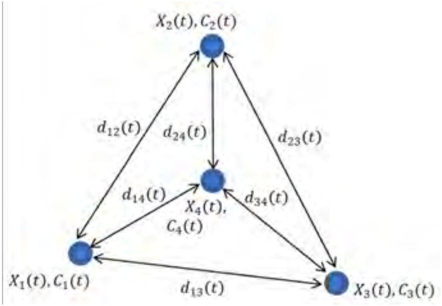

Figure: Illustration using 4 Nodes.

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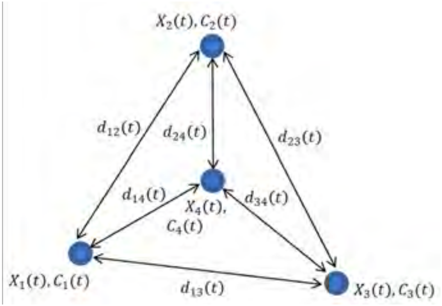

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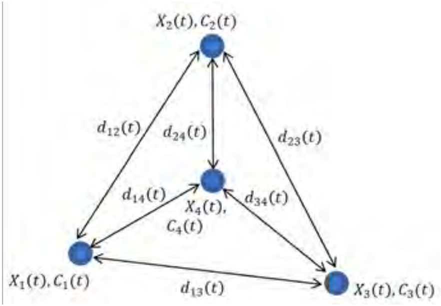

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Apps.: Situational awareness, Interferometry, Relative inference and control, ⋯

Problem statement: Timing and Localization

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- Consider a set of $\mathcal{N} = \{1, 2, \cdots, N\}$ cooperative mobile nodes, in D dimensional space
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Relative Timing

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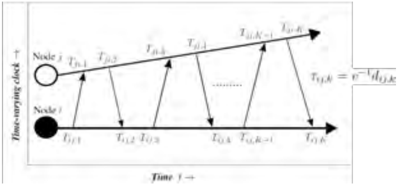

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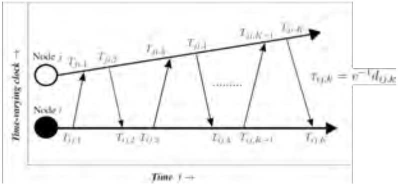

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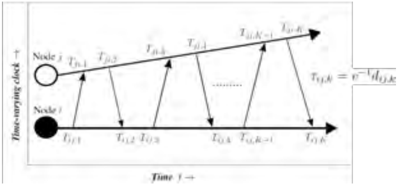

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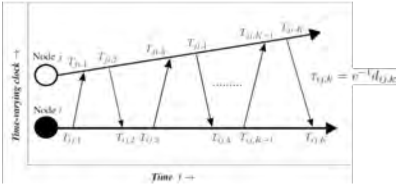

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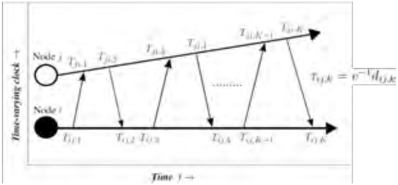

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$$\min_{\phi_j, d_{ij}} \|\mathcal{C}_i(\phi_i, T_{ij,k}) - \mathcal{C}_j(\phi_j, T_{ji,k}) \pm \tau_{ij}\| \quad (1)$$

- Solution: Least squares, BLUE, MLE,...

Freris et.al., (2010) Fundamental limits on synchronizing clocks over networks

Generalized Two-Way Ranging (GTWR)

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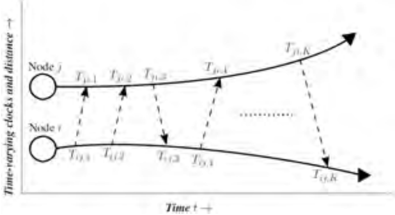

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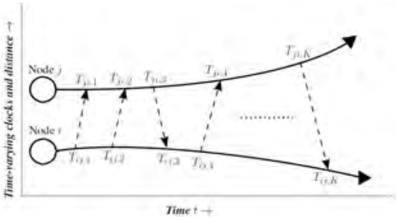

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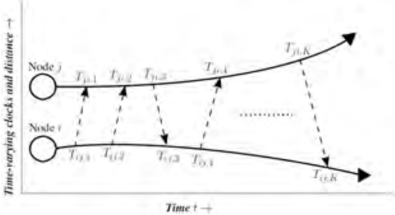

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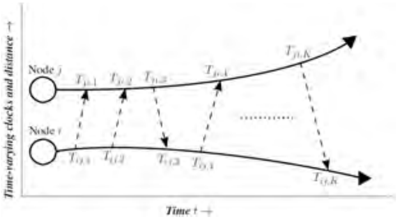

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Figure: Timing

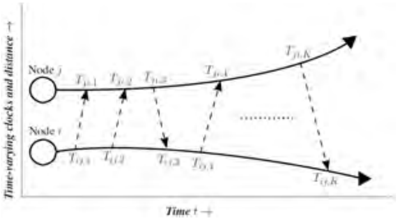

Figure: Generalized Two-way Ranging (GTWR)

- Let $\mathcal{C}_j(\boldsymbol{\phi}_j, T_{ij,k})$ be the local clock model at the jth node, where $\boldsymbol{\phi}_j = \{\phi_j, \dot{\phi}_j\}$
- Time-varying distance between nodes i.e., $\tau_{ji,k} = \tau_{ij,k}, \tau_{ij,k} = c^{-1} d_{ij,k} \;\forall\; k \in \mathcal{K}$
- Trick: Approximate distance $d_{ij,k}$ as a Taylor polynomial w.r.t. time
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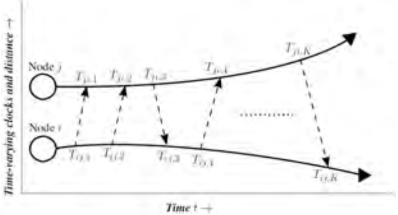

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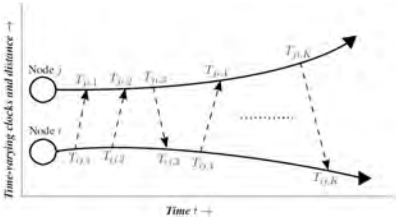

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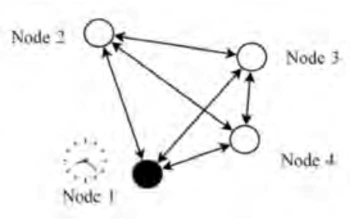
  where $\mathbf{e}_{ij}$ is a direction indication vector and $G_{ij}(\cdot)$ is a function of space-time.
- Challenge: Ill-posed problem, since a reference is needed to resolve ambiguity.

Rajan et.al., (2015) Joint ranging and synchronization for an anchorless network of mobile nodes

Reference-free Synchronization



- Single clock reference
- Multiple clock references

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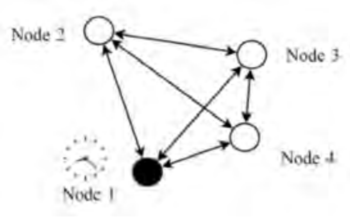

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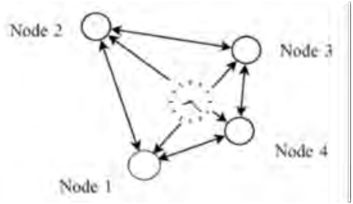

- Sum based clock reference
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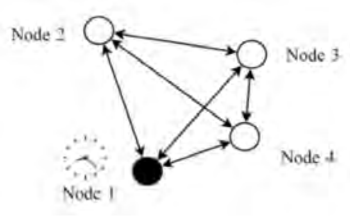

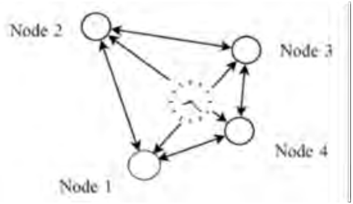

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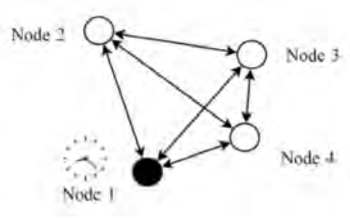

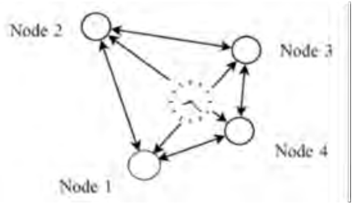

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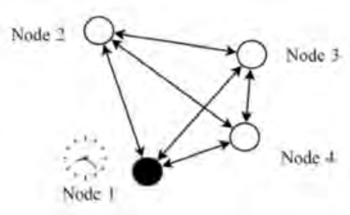

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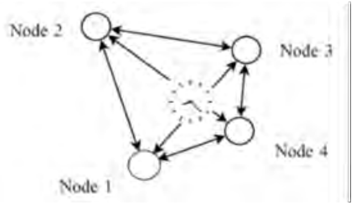

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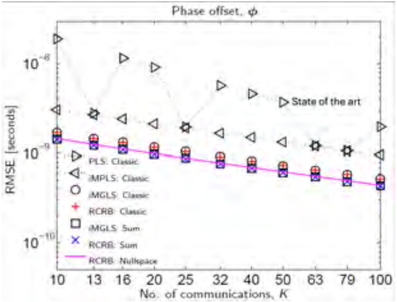
- Derive lower bounds (e.g., CRLB) and compare with state of the art methods

Rajan et.al., (2015) Joint ranging and synchronization for an anchorless network of mobile nodes

Simulations: Reference-free Synchronization



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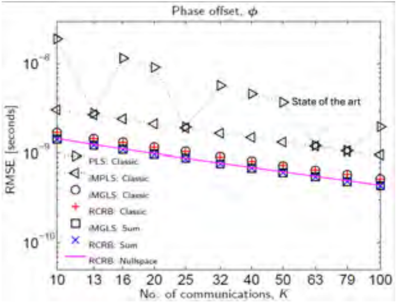

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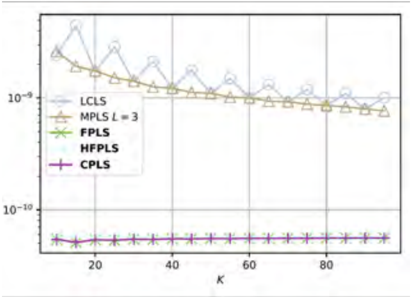

Figure: Frequency-augmented synchronization

Rajan et.al., (2015) Joint ranging and synchronization for an anchorless network of mobile nodes
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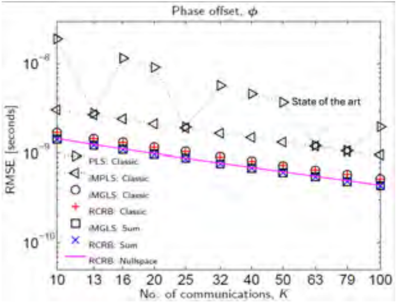

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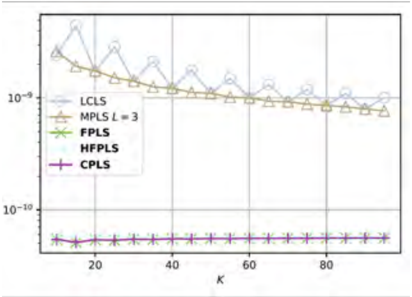

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How much improvement does *sum* reference offer compared to any arbitrary reference ?

Simplified clock model

Consider a network of N clocks, where each collects $K \geq N$ measurements

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Extending the model for all N sensors we have

$$\mathbf{y} = \mathbf{s} + (\mathbf{I}_N \otimes \mathbf{1}_K)\boldsymbol{\theta} + \boldsymbol{\eta} = \mathbf{s} + \mathbf{H}\boldsymbol{\theta} + \boldsymbol{\eta} \tag{4}$$

where

$$\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T, \cdots, \mathbf{y}_N^T]^T \quad , \quad \mathbf{s} = [\mathbf{s}_1^T, \mathbf{s}_2^T, \cdots, \mathbf{s}_N^T]^T$$

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Given the measurements $\mathbf{y}$, and unknown $\mathbf{s}$,

- Can we estimate the unknown $\boldsymbol{\theta}$ without an external reference ?
- What is an optimal estimator (or reference) for such a problem ?

Blind clock synchronization

- Let $\mathbf{S} = [\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_N]$ be a $K \times N$ matrix, which lies in an $r-$dim. subspace $r < N$

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Clearly minimizing $\|\bar{\mathbf{\Gamma}}\mathbf{H}\boldsymbol{\theta}\|_2$ in (6) yields solutions spanning the nullspace of $\mathbf{\Gamma}$

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Cramér Rao Lower Bound theorem (CRB)

Theorem (Rao1945 [2])

Consider the estimation of an unknown parameter $\hat{\boldsymbol{\theta}} = g(\mathbf{y})$*, where the data* $\mathbf{y}$ *is drawn from a pdf* $p(\mathbf{y};\boldsymbol{\theta})$*, which satisfies the regularity condition:*

$$\mathbb{E}\left[\frac{\partial \ln\ p(\mathbf{y};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right] = \mathbf{0},$$

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The FIM for our model $\bar{\boldsymbol{\Gamma}}\mathbf{y} \sim \mathcal{N}\left(\bar{\boldsymbol{\Gamma}}\mathbf{H}\boldsymbol{\theta}, \bar{\boldsymbol{\Gamma}}(\boldsymbol{\Sigma} \otimes \mathbf{I}_K)\bar{\boldsymbol{\Gamma}}^T\right)$ is $\mathbf{F} = K\boldsymbol{\Gamma}^T(\boldsymbol{\Gamma}\boldsymbol{\Sigma}\boldsymbol{\Gamma}^T)^{-1}\boldsymbol{\Gamma}$

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Constrained Cramér Rao Lower Bound (CCRB)

Theorem (Stoica1998[3])

Consider a consistent set of k continuously differentiable constraints on $\hat{\boldsymbol{\theta}}$ i.e., $c(\hat{\boldsymbol{\theta}}) = 0$, and let $\mathbf{C}(\boldsymbol{\theta}) = \partial \mathbf{c}(\boldsymbol{\theta}) / \partial \boldsymbol{\theta}^T$ be the gradient matrix which is full row rank, then the Constrained Cramér Rao lower Bound (CCRB) on the variance of any unbiased estimator exists and is bounded by

$$\mathbb{E}\left\{(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T\right\} \equiv \boldsymbol{\Sigma}_{\hat{\theta}} \geq \mathbf{U}\left(\mathbf{U}^T \mathbf{F} \mathbf{U}\right)^{-1} \mathbf{U}^T, \tag{7}$$

where $\mathbf{U}$ spans the null space of the gradient matrix $\mathbf{C}(\boldsymbol{\theta})$.

[3] **Stoica; P. and Ng; B.C.; 1998. On the Cramr-Rao bound under parametric constraints. IEEE Signal Processing Letters; 5(7); pp.177-179.**

[4] **Rajan; R.T. and van der Veen; A.-J; 2015; Joint ranging and synchronization for an anchorless network of mobile nodes**

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Consider a consistent set of k continuously differentiable constraints on $\hat{\boldsymbol{\theta}}$ i.e., $c(\hat{\boldsymbol{\theta}}) = 0$, and let $\mathbf{C}(\boldsymbol{\theta}) = \partial \mathbf{c}(\boldsymbol{\theta}) / \partial \boldsymbol{\theta}^T$ be the gradient matrix which is full row rank, then the Constrained Cramér Rao lower Bound (CCRB) on the variance of any unbiased estimator exists and is bounded by

$$\mathbb{E}\left\{(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T\right\} \equiv \boldsymbol{\Sigma}_{\hat{\theta}} \geq \mathbf{U}\left(\mathbf{U}^T \mathbf{F} \mathbf{U}\right)^{-1} \mathbf{U}^T, \tag{7}$$

where $\mathbf{U}$ spans the null space of the gradient matrix $\mathbf{C}(\boldsymbol{\theta})$.

Lemma (Optimal constraint[4])

Any set of vectors spanning the nullspace of the FIM is an optimal constraint set.

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Data-driven reference: For our model, $\mathbf{F} = K\boldsymbol{\Gamma}^T(\boldsymbol{\Gamma}\boldsymbol{\Sigma}\boldsymbol{\Gamma}^T)^{-1}\boldsymbol{\Gamma}$, if i.e., $K \geq N$, and if the covariance $\boldsymbol{\Sigma}$ is known, then an optimal $\mathbf{C}(\boldsymbol{\theta})$ can be designed.

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How much improvement does (11) offer as compared to Constraint 1 ?

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Observe:

$$\delta \equiv \frac{\mathrm{Tr}(\boldsymbol{\Sigma}_2)}{\mathrm{Tr}(\boldsymbol{\Sigma}_1)} = \frac{1}{2} \tag{14}$$

Key results

Theorem (Optimal reference [5])

Consider an network of N clocks, where each clock collects K measurements based on the data model (8), then the following statements hold.

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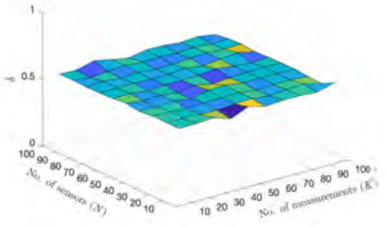

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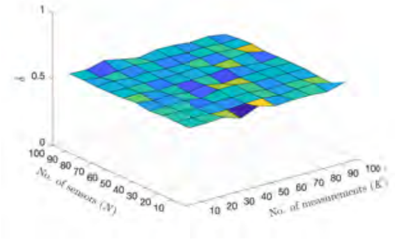

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Ongoing work: Effect of Bias, Generalization to Bayesian Gauss Markov Models, PCRB, ⋯

Relative Localization

Problem statement: Relative localization

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Note: This idea can be extended to higher-order kinematics

Euclidean Distance Matrix

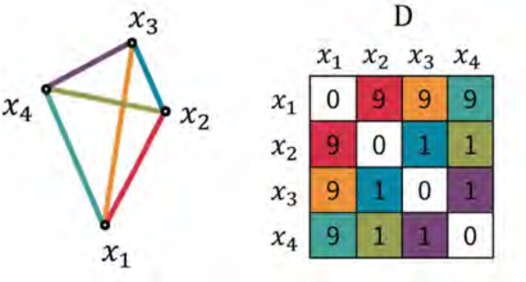

Figure: Illustration of a Euclidean Distance Matrix with $N = 4$ nodes

Euclidean Distance Matrices (EDM): $\mathbf{D} = [d_{ij}] \in \mathbb{R}^{N \times N}$

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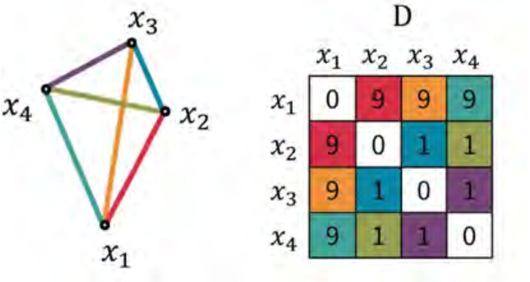

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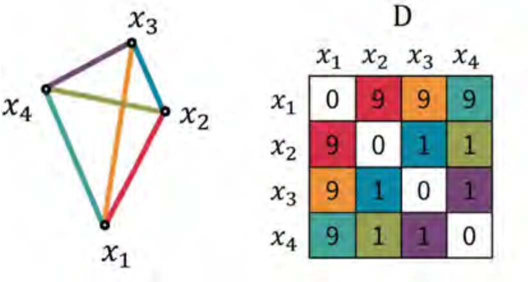

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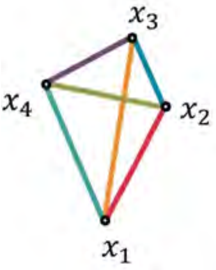

	x_1	x_2	x_3	x_4
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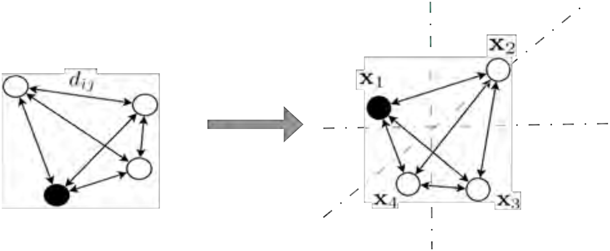

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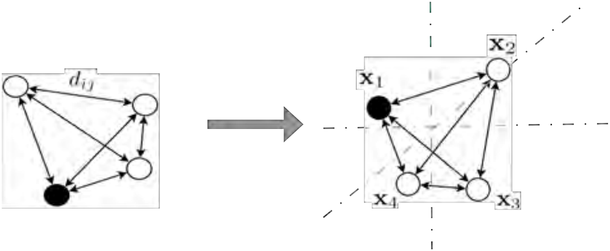

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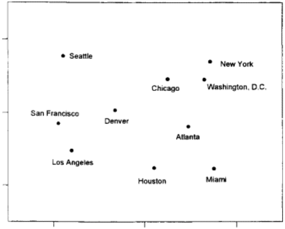
Multi-Dimensional Scaling (MDS)



City 1	2	3	4	5	6	7	8	9	10
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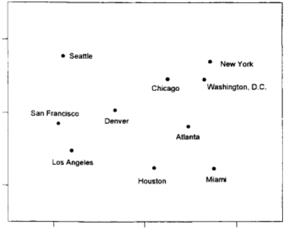

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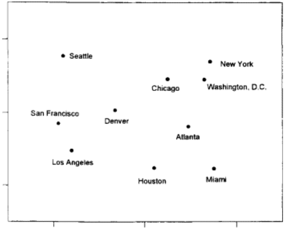

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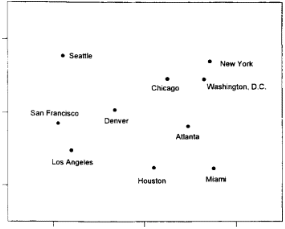

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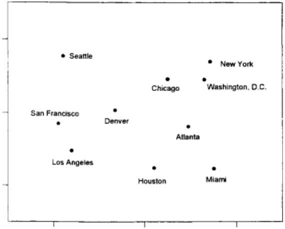

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Torgerson, Warren S. "Multidimensional scaling: I. Theory and method." Psychometrika 17.4 (1952)

Relative localization and velocity

Time-varying distances to Relative kinematics

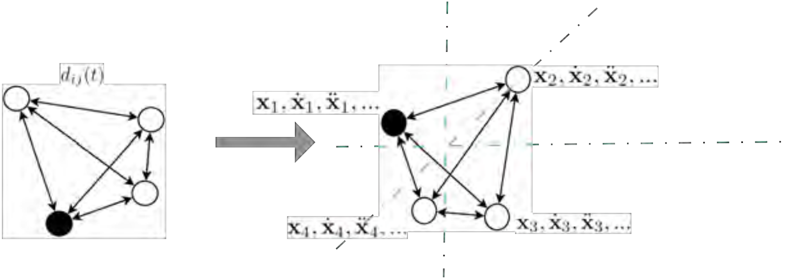

Figure: Time-varying Euclidean Distances to Relative kinematics

- Time-varying Euclidean Distance Matrices (EDM): $\mathbf{D}(t) = [d_{ij}(t)] \in \mathbb{R}^{N \times N}$
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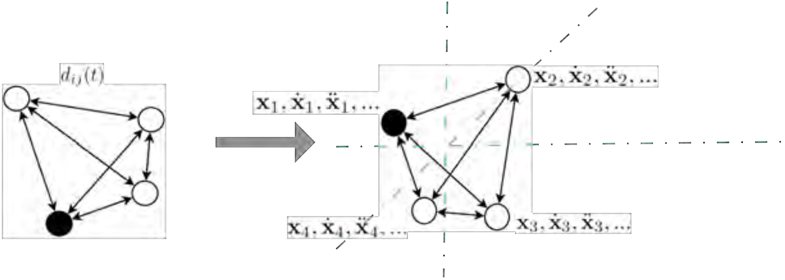

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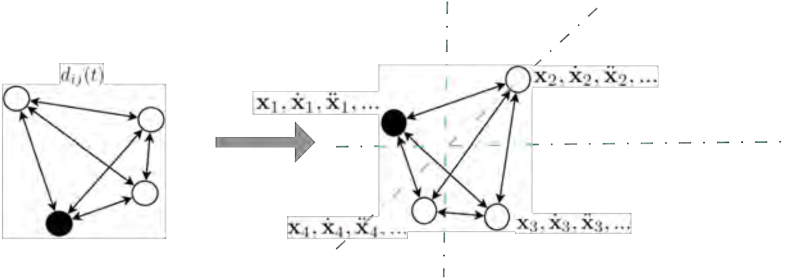

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$$\tau_{ij,k} = c^{-1} d_{ij,k}$$

$$T_{ji,k} = \begin{cases} T_{ij,k} + c^{-1} d_{ij,k} & \text{for } i \rightarrow j \\ T_{ij,k} - c^{-1} d_{ij,k} & \text{for } i \leftarrow j \end{cases}$$

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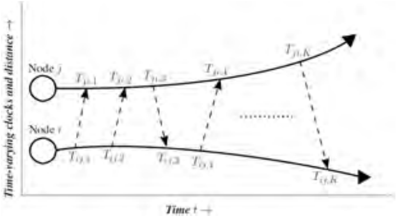

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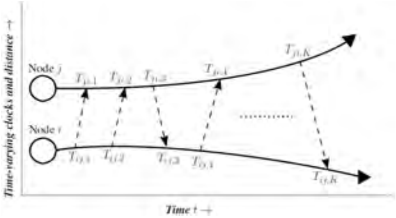

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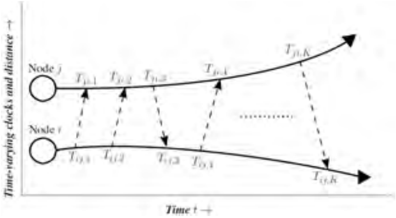

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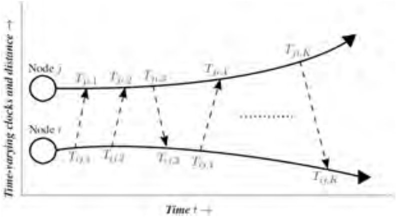

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$$\text{(a)}\ \mathbf{B}_0 \triangleq \underline{\mathbf{B}}^{(0)} = \underline{\mathbf{X}}^T\underline{\mathbf{X}} = -0.5\mathbf{P}\mathbf{R}^{\odot 2}\mathbf{P},$$

Linearized Multi-dimensional Scaling

Consider a first-order motion: $\underline{\mathbf{S}}(t) = \underline{\mathbf{X}} + \mathbf{H}_1\underline{\mathbf{Y}}_1 t$

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(b) Solve for relative velocities $\hat{\mathbf{Y}}_1 = \arg\min_{\mathbf{Y}_1} \|\hat{\mathbf{B}}_2 - \underline{\mathbf{Y}}_1^T\underline{\mathbf{Y}}_1\|$ s.t. $\text{rank}(\mathbf{Y}_1) = D$

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Simulations results $N = 10$ nodes

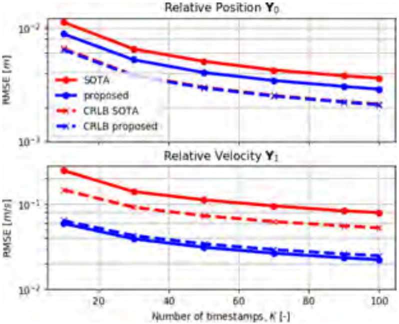

Figure: Constant Velocity

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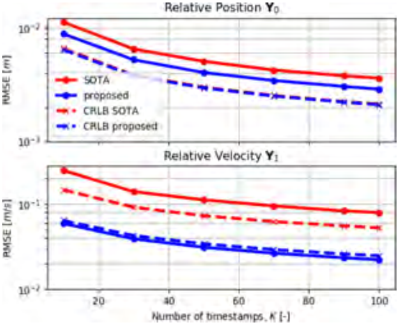

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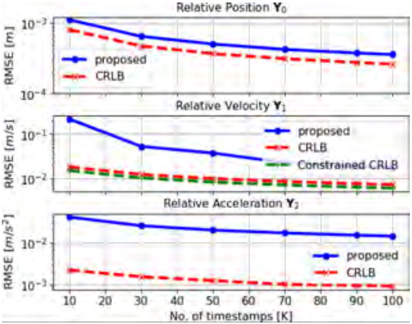

Figure: Constant Acceleration

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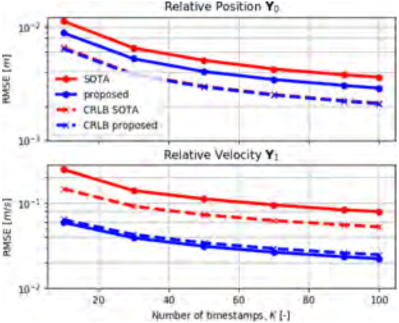

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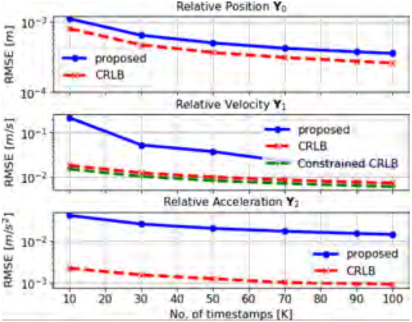

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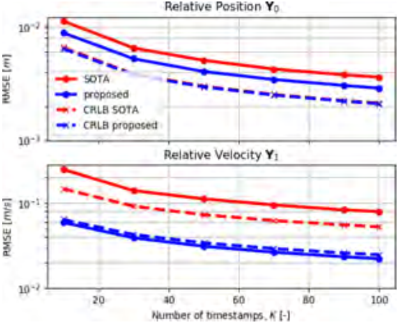

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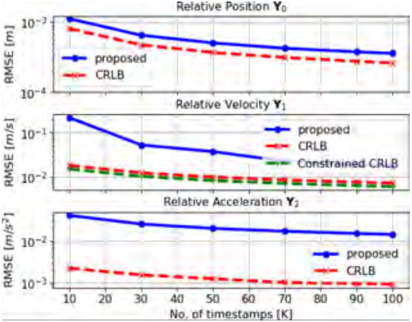

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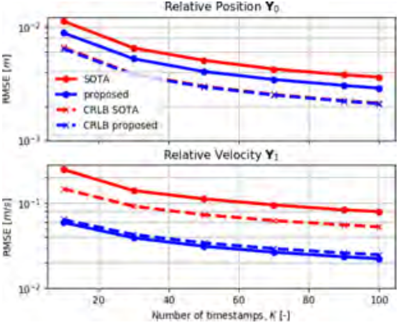

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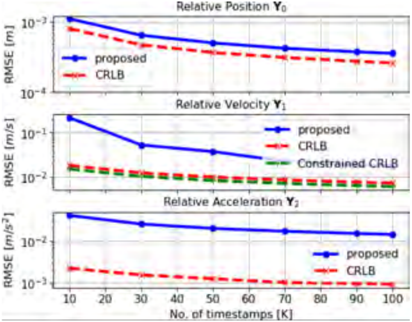

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- Lyapunov-like equation : $\mathbf{A}^T \mathbf{X} + \mathbf{X}^T \mathbf{A} = \mathbf{B}$

Braden, H. W. (1998). The equations $A^T X \pm X^T A = B$. SIAM Journal on Matrix Analysis and Applications
Rajan et.al., (2025) The equations $\mathbf{A}^T \mathbf{X} + \mathbf{X}^T \mathbf{A} = \mathbf{B}$ under transformations (in submission)

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Thank you for your attention !