More than Positive Weights: Structural Balance and Random Walks

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Overview

Background

Signed networks
- Structural characterisation
- Dynamical characterisation
- Experiments

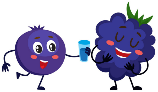
Complex-weighted networks

About me: PhD research

InFoMM Industrially Focused Mathematical Modelling

TESCO

- Retail industry: huge turnover but small margins.
- Product relationships: **complements** and **substitutes**.

- Further decision making: product-catalogue design, store layout, stock levels, promotions $\longleftarrow$ **demand dynamics**.




Research interest: Network science

- Structural properties: clustering / community detection on networks, spectral properties via adjacency matrix and graph Laplacian.
- Dynamical properties: linear dynamics such as random walks, and nonlinear dynamics such as linear threshold models.

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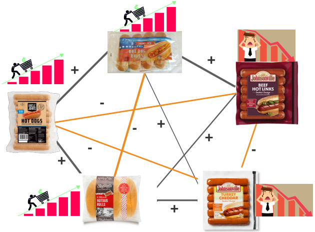
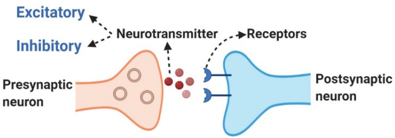
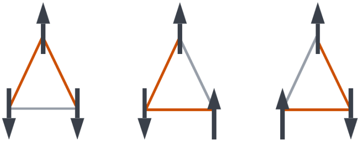


→ Signed networks

Signed networks

What if there are negative connections?





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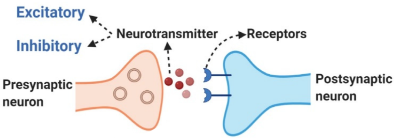

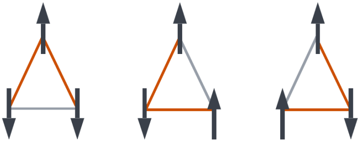

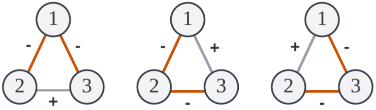

Structural balance

A specific type of signed networks that are relatively stable.

- Motivation (Harary, 1953; Cartwright and Harary, 1956): “friend of a friend is a friend, enemy of a friend is an enemy, while enemy of an enemy is a friend“.
- Mathematical interpretation: no cycle with an odd number of negative edges, which defines the cycle to be “negative”.

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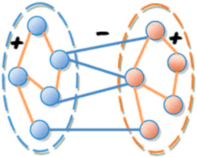
- Structural Theorem for Balance (Harary, 1953): $G = (V, E)$ is structurally balanced $\Leftrightarrow$ $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$ *s.t.* any edge within each node subset is positive while any edge between the two node subsets is negative.

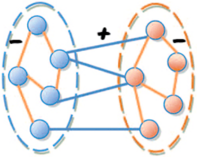


Structural antibalance

The opposite to structural balance.

- Definition: no cycle with an odd number of positive edges.

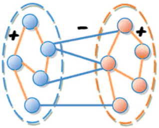
- Structural Theorem for Antibalance (Harary, 1957): $G = (V, E)$ is structurally antibalanced $\Leftrightarrow$ $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$, *s.t.* any edge within each node subset is negative while any edge between the two node subsets is positive.



Classification

Structurally balanced, structurally antibalanced, and strictly unbalanced signed networks.

- Strictly unbalanced: if it is neither balanced nor antibalanced.



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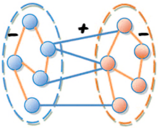
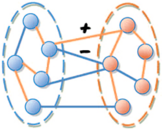


Structurally antibalanced



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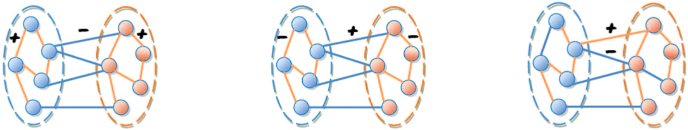

Structurally balanced   Structurally antibalanced   Strictly unbalanced

$\rightarrow$ Characterisation of weighted signed networks $G = (V, E, \mathbf{W})$:

- signed weight matrix $\mathbf{W} = (W_{ij}) \in \mathbb{R}^{n \times n}$, where $n = |V|$ and $W_{ij} \neq 0$ if $(v_i, v_j) \in E$;
- unsigned weights matrix $\bar{\mathbf{W}} = (\bar{W}_{ij}) \in (\mathbb{R}_+ \cup \{0\})^{n \times n}$, where $\bar{W}_{ij} > 0$ if $(v_i, v_j) \in E$.

Spectral properties

Spectral properties of the weight matrix.

Theorem (Spectral Theorem for Balance and Antibalance)

Considering the unitary decompositions of the signed weight matrix $\mathbf{W} = \mathbf{U}\Lambda\mathbf{U}^T$, and the unsigned one $\bar{\mathbf{W}} = \bar{\mathbf{U}}\bar{\Lambda}\bar{\mathbf{U}}^T$:

1. Structurally balanced: $\Lambda = \bar{\Lambda}$, $\mathbf{U} = \mathbf{I}_1\bar{\mathbf{U}}$.
2. Structurally antibalanced: $\Lambda = -\bar{\Lambda}$, $\mathbf{U} = \mathbf{I}_1\bar{\mathbf{U}}$.

$\mathbf{I}_1$ denote the diagonal matrix whose (i, i) element is 1 if $i \in V_1$ and -1 otherwise, where V_1, V_2 denote the corresponding bipartition for either balanced or antibalanced networks

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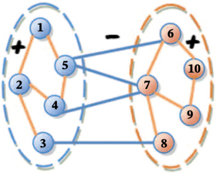

$$\mathbf{I}_1 = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \\ & & & -1 & & \\ & 0 & & & \ddots & \\ & & & & & -1 \end{bmatrix} \begin{matrix} \mathbf{1} \\ \vdots \\ \mathbf{5} \\ \mathbf{6} \\ \vdots \\ \mathbf{10} \end{matrix}$$

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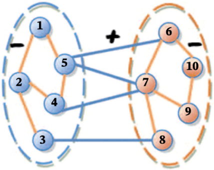

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Proof ideas:

1. for balanced networks, $\mathbf{W} = \mathbf{I}_1\bar{\mathbf{W}}\mathbf{I}_1$;
2. for antibalanced networks, $\mathbf{W} = -\mathbf{I}_1\bar{\mathbf{W}}\mathbf{I}_1$.

Spectral properties: Strictly unbalanced networks

When the signed network is neither balanced nor antibalanced.

Theorem (Spectral Theorem for Strict Unbalance)

A signed network G is strictly unbalanced if and only if $\rho(\mathbf{W}) < \rho(\bar{\mathbf{W}})$, where $\rho(\mathbf{W}) = \max\{|\lambda_i| : \lambda_i \text{ is an eigenvalue of } \mathbf{W}\}$.

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Proof ideas:

- If G is either balanced or antibalanced, $\rho(\mathbf{W}) = \rho(\bar{\mathbf{W}})$.
- Lemma: If G is strictly unbalanced, we can find two walks between nodes $v_i, v_j \in V$ of the same length but of different signs.
- The previous conflict will contract the spectral radius via the definition $\rho(\mathbf{W}) = ||\mathbf{W}||_2 = \max_{||\mathbf{x}||_2=1} ||\mathbf{W}\mathbf{x}||_2$.

Dynamics: Random walks

Extension to signed networks: polarisation on each node.

- Weight matrix: $\mathbf{W} = \mathbf{W}^{+} - \mathbf{W}^{-}$, where $W_{ij}^{+} = |W_{ij}|$ if $W_{ij} > 0$ (0 otherwise), and $W_{ij}^{-} = |W_{ij}|$ if $W_{ij} < 0$ (0 otherwise).

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- State values: $x_j^+(t), x_j^-(t)$ as density of positive, negative walkers, resp.

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  - $x_j^+(t) = \sum_i \frac{1}{d_i} \left(W_{ij}^+ x_i^+(t-1) + W_{ij}^- x_i^-(t-1) \right)$;

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$$x_j(t) = \sum_i \frac{1}{d_i}\left(W_{ij}^+ - W_{ij}^-\right)\left(x_i^+(t-1) - x_i^-(t-1)\right) = \sum_i \frac{1}{d_i} W_{ij} x_i(t-1).$$

Hence, $\mathbf{x}(t) = \mathbf{P}\mathbf{x}(t-1)$, where $\mathbf{P} = \mathbf{D}^{-1}\mathbf{W}$, $\mathbf{D} = \mathbf{Diag}(\mathbf{d})$, and $\mathbf{d} = (d_i)$.

Dynamics: Random walks

Structurally balanced signed networks.

Proposition

$\mathbf{P}$ has eigenvalue 1 if and only if G is balanced.

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Proof ideas:

- $\mathbf{P} = \mathbf{D}^{-1/2}\mathbf{P}_{sym}\mathbf{D}^{1/2}$, where $\mathbf{P}_{sym} = \mathbf{D}^{-1/2}\mathbf{W}\mathbf{D}^{-1/2}$.
- $\lambda_{max}(\bar{\mathbf{P}}_{sym}) = 1$; Spectral Theorems.

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Proposition

If G is balanced and is not bipartite, then the steady state is $\mathbf{x}^* = (x_j^*)$ where

$$x_j^* = \begin{cases} (\mathbf{x}(0)^T \mathbf{I}_1 \mathbf{1}) d_j/(2m), & \text{if } v_j \in V_1, \\ -(\mathbf{x}(0)^T \mathbf{I}_1 \mathbf{1}) d_j/(2m), & \text{otherwise,} \end{cases}$$

where $2m = \sum_j d_j$, and $\mathbf{1}$ is the all-one vector.

Dynamics: Random walks

Structurally antibalanced signed networks.

Proposition

$\mathbf{P}$ has eigenvalue -1 if and only if G is antibalanced.

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Proposition

If G is antibalanced and is not bipartite, then the random walks have different limits for odd or even times, denoted by $\mathbf{x}^{*o} = (x_j^{*o})$ and $\mathbf{x}^{*e} = (x_j^{*e})$, respectively, where

$$x_j^{*o} = \begin{cases} -(\mathbf{x}(0)^T \mathbf{I}_1 \mathbf{1}) d_j/(2m), & \text{if } v_j \in V_1, \\ (\mathbf{x}(0)^T \mathbf{I}_1 \mathbf{1}) d_j/(2m), & \text{otherwise,} \end{cases}$$

while

$$x_j^{*e} = \begin{cases} (\mathbf{x}(0)^T \mathbf{I}_1 \mathbf{1}) d_j/(2m), & \text{if } v_j \in V_1, \\ -(\mathbf{x}(0)^T \mathbf{I}_1 \mathbf{1}) d_j/(2m), & \text{otherwise.} \end{cases}$$

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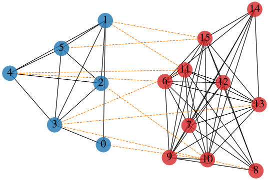
- Further characterisation:
  - "Distance" from being balanced: $d_b(G) = -(\lambda_{\max}(\mathbf{P}(G)) - 1)$;
  - "Distance" from being antibalanced: $d_a(G) = \lambda_{\min}(\mathbf{P}(G)) - (-1)$,

$\propto$ #edges disturbing the balanced or antibalanced structure, by perturbation analysis.

Numerical experiments

Different types of signed networks.



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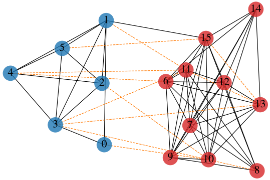
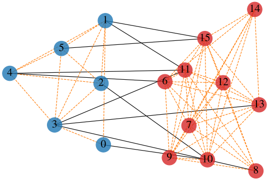
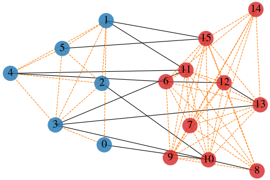


Strictly unbalanced (close to balanced)



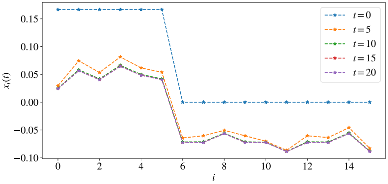
Structurally antibalanced



Strictly unbalanced (close to antibalance)

Numerical experiments

Evolution of the state values from the signed random walks.



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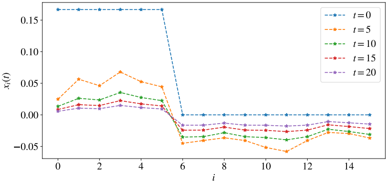

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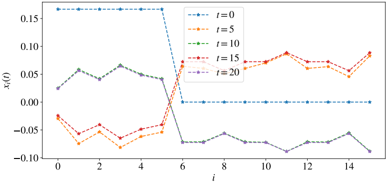

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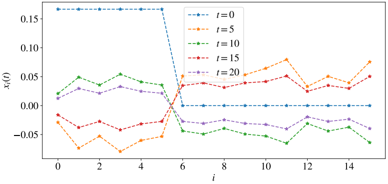

Strictly unbalanced (close to antibalance)

Summary

Spreading and structural balance on signed networks.

- Structural properties: (i) classification based on structural balance, and (ii) characterisation of the spectral properties.
- Dynamical properties: characterisation of random walks in each type of signed networks.

Future directions

- Applications to various fields.
- More switching equivalence classes.

Main references:

YT, and R. Lambiotte. Spreading and structural balance on signed networks. *SIAM J. Appl. Dyn. Syst.*, Accepted, 2023.

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Background

Signed networks
Structural characterisation
Dynamical characterisation
Experiments

Complex-weighted networks

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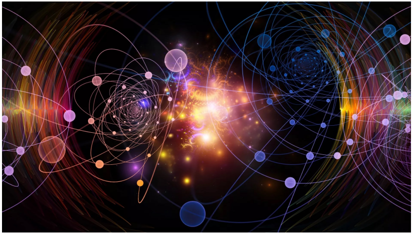
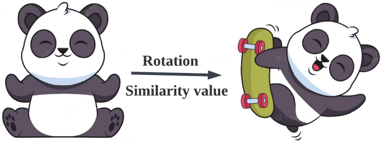
What if negative connections are not enough?





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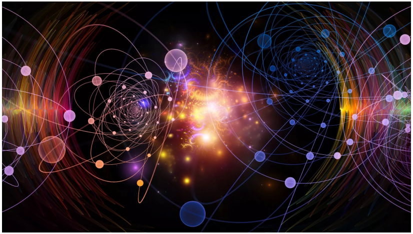

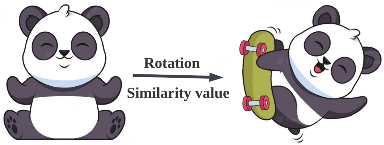

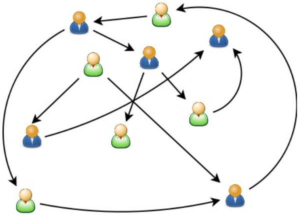

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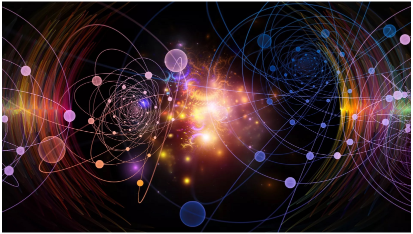

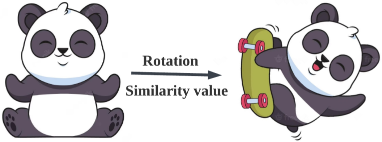

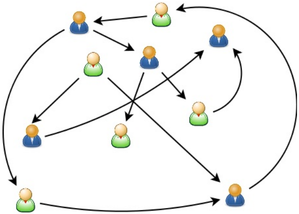

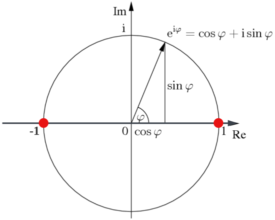

Structural balance

A specific type of complex-weighted networks that are relatively stable.

- Phase of cycles: sum of phases of composing edges.
- Structural balance: all cycles have phase 0 (up to multiples of 2π).

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- Structural Theorem for Balance: $G(V, E)$ is structurally balanced $\Leftrightarrow$ partition $\{V_i\}_{i=1}^{l_p}$ s.t.
  - any edges within have phase 0,
  - any edges between the same pair of node subsets have same phase,
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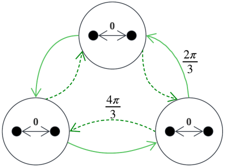

Structural antibalance

The opposite to structural balance.

- Definition: all cycles, after adding π to their composing edges, have phase 0 (up to multiples of 2π).

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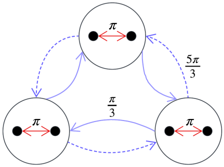

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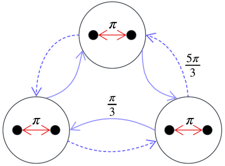

- Strictly unbalanced: if it is neither balanced nor antibalanced.

Spectral properties

Spectral properties of the weight matrix.

Theorem (Spectral Theorem for Balance and Antibalance)

Considering the unitary decompositions of the complex weight matrix $\mathbf{W} = \mathbf{U}\Lambda\mathbf{U}^*$, and the one ignoring the phase $\bar{\mathbf{W}} = \bar{\mathbf{U}}\bar{\Lambda}\bar{\mathbf{U}}^*$:

1. Structurally balanced: $\Lambda = \bar{\Lambda}$, $\mathbf{U} = \mathbf{I}_1\bar{\mathbf{U}}$.
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$\mathbf{I}_1$ denote the diagonal matrix whose (i, i) element is $\exp\big(\mathrm{i}\theta_{1\sigma(i)}\big)$, where $\sigma(\cdot)$ returns the node subset index, and θ_{hl} is the phase from V_h to V_l (balance) and is the phase after adding π to each composing edge (antibalance).

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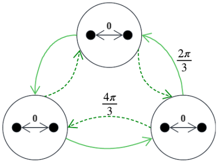

$$\mathbf{I}_1 = \begin{bmatrix} 1 & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & \exp\left(\mathrm{i}\frac{2\pi}{3}\right) & & & & & \\ & & & & \ddots & & & & \\ & & & & & \exp\left(\mathrm{i}\frac{2\pi}{3}\right) & & & \\ & & & & & & \exp\left(\mathrm{i}\frac{4\pi}{3}\right) & & \\ & & & & & & & \ddots & \\ 0 & & & & & & & & \exp\left(\mathrm{i}\frac{4\pi}{3}\right) \end{bmatrix}$$

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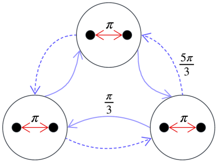

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Spectral properties: Strictly unbalanced networks

When the signed network is neither balanced nor antibalanced.

Theorem (Spectral Theorem for Strict Unbalance)

A complex-weighted network G is strictly unbalanced if and only if $\rho(\mathbf{W}) < \rho(\bar{\mathbf{W}})$, where $\rho(\mathbf{W}) = \max\{|\lambda_i| : \lambda_i \text{ is an eigenvalue of } \mathbf{W}\}$.

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Proof ideas:

- If G is either balanced or antibalanced, $\rho(\mathbf{W}) = \rho(\bar{\mathbf{W}})$.
- Lemma: If G is strictly unbalanced, we can find two walks between nodes $v_i, v_j \in V$ of the same length but of different phases.
- The previous conflict will contract the spectral radius via the definition $\rho(\mathbf{W}) = ||\mathbf{W}||_2 = \max_{||\mathbf{x}||_2=1} ||\mathbf{W}\mathbf{x}||_2$.

Dynamics: Random walks

Extension to complex-weighted networks.

- Weight matrix: $\mathbf{W} = \int_0^{2\pi} e^{\mathrm{i}\theta}\mathbf{W}^\theta d\theta$, where $\mathbf{W}^\theta = (W_{ij}^\theta)$ with $W_{ij}^\theta = |W_{ij}|\delta(\theta - \varphi_{ij})$ encoding the presence of an edge with phase θ.

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Hence, $\mathbf{x}(t+1) = \mathbf{P}\mathbf{x}(t)$, where $\mathbf{P} = \mathbf{D}^{-1}\mathbf{W}$, $\mathbf{D} = \mathbf{Diag}(\mathbf{d})$, and $\mathbf{d} = (d_i)$.

Dynamics: Random walks

Dynamical properties in a nutshell.

- Structural balance: **P** has eigenvalue 1, and if G is not bipartite, the steady state is $\mathbf{x}^* = (x_j^*)$,

$$x_j^* = \exp\big(\theta_{1\sigma(j)}\mathrm{i}\big)(\mathbf{x}(0)^*\mathbf{l}_1^*\mathbf{1})d_j/(2m),$$

where $2m = \sum_j d_j$, and $\mathbf{1}$ is the all-one vector.

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- Strict unbalance: $\rho(\mathbf{P}) < 1$, and the steady state is $\mathbf{0}$, the vector of zeros.

Summary

Structural balance and random walks on complex networks with complex weights.

- Structural properties: (i) classification based on structural balance, and (ii) characterisation of the spectral properties.
- Dynamical properties: extension and characterisation of random walks in each type of complex-weighted networks.

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Applications

- Spectral clustering.
- Magnetic Laplacian.

Main references:

YT, and R. Lambiotte. Structural balance and random walks on complex networks with complex weights. *arXiv*, arXiv:2307.01813, 2023.

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