On mixed network coordination/anticoordination games

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Focus period

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Outline

1. Motivation and problem description
2. Related work and main issues
3. Results
4. Conclusions and extensions

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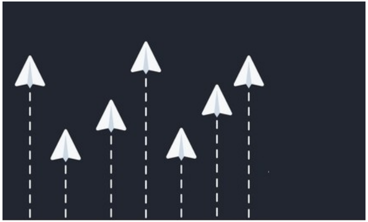
- Network **coordination** games:
  - strategic complements effects
  - adoption of beliefs or behavioral attitudes, spread of a new technology (Morris, 2000), (Montanari and Saberi, 2010)



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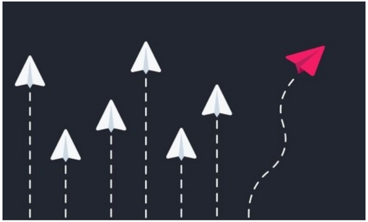
  - competition for resources, gains from differentiation (Bramoullé 2007), (Monaco and Sabarwal, 2016)



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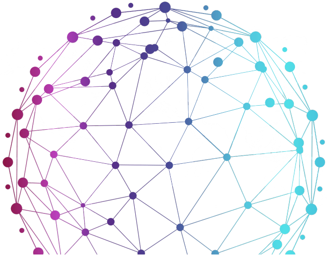
- Mixed network **coordination/anti-coordination (CAC)** games:
  - a model of heterogeneous interactions over a network system
  - coexisting coordinating and anti-coordinating agents (Grabish and Li, 2019), (Ramazi et all, 2023)



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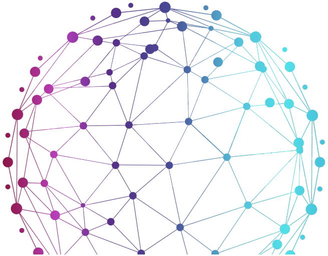


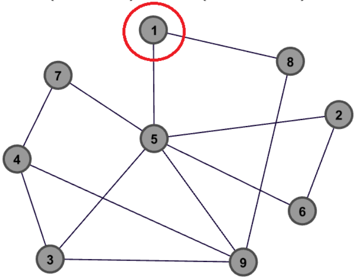
Relevance of the **topological structure** of the graph in determining existence and structure of equilibrium configurations and dynamical properties of the network systems

Network games (Jackson and Zenou, 2015)

Models for strategic interactions over interconnected systems

- **Game**: $(\mathcal{V}, \mathcal{A}, \{u_i\}_{i\in\mathcal{V}})$
  - Agent set: $\mathcal{V}$
  - Action set: $\mathcal{A}$
  - Utilities: $u_i : \mathcal{A}^{\mathcal{V}} \to \mathbb{R}$, $i \in \mathcal{V}$
- **Network game**
  - Agent set coincides with node set of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$
  - Utilities depend only on their action and their **neighbors**' actions

$u_1(x_1, x_{-1}) = u_1(x_1, x_5, x_8)$



Mixed network coordination anti-coordination (CAC) game

- Finite agent set $\mathcal{V} = \mathcal{R} \cup \mathcal{S}$
  - $\mathcal{R}$ coordinating agents
  - $\mathcal{S}$ anti-coordinating agents

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⋆ Agent types $\delta_i = \begin{cases} 1 & \text{if } i \in \mathcal{R} \\ -1 & \text{if } i \in \mathcal{S} \end{cases}$

⋆ Agent weights $d_i \in \mathbb{R}$

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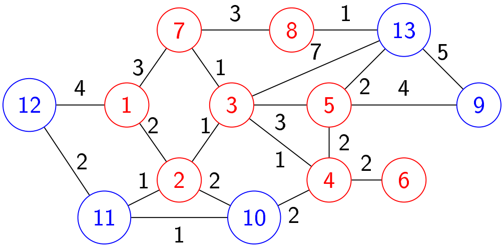
- Undirected **graph** structure $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$
  - W: non-negative, symmetric, no self loops



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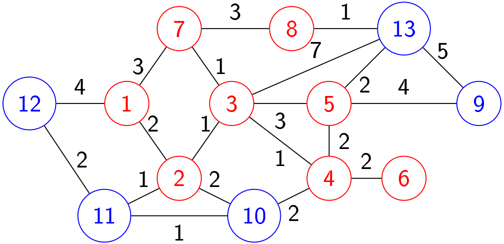

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- Binary action set $\mathcal{A} = \{-1, +1\}$

Utility $i \in \mathcal{V}$**:**

$$u_i(x_i, x_{-i}) = \delta_i \left(\sum_{j \in \mathcal{V}} W_{ij} x_i x_j + d_i x_i \right)$$

$x_i \in \mathcal{A}$ action of i and $x_{-i} \in \mathcal{A}^{\mathcal{V} \setminus \{i\}}$ actions of others

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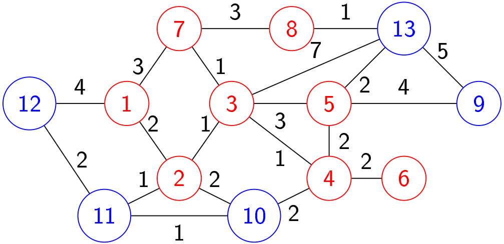

Utility $i \in \mathcal{R}$: $$u_i(x_i, x_{-i}) = \sum_{j \in \mathcal{V}} W_{ij} x_i x_j + d_i x_i$$

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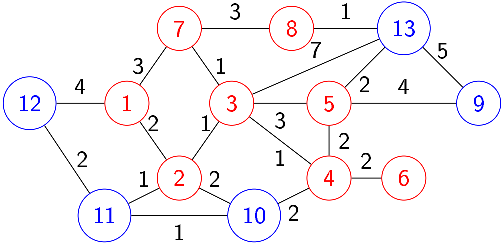

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Utility $i \in \mathcal{S}$:

$$u_i(x_i, x_{-i}) = -\sum_{j \in \mathcal{V}} W_{ij} x_i x_j - d_i x_i$$

$x_i \in \mathcal{A}$ action of i and $x_{-i} \in \mathcal{A}^{\mathcal{V} \setminus \{i\}}$ actions of others

Nash Equilibria: existence and reachability

- Best-response function

$$\mathcal{B}_i(x_{-i}) = \arg\max_{x_i \in \mathcal{A}} u_i(x_i, x_{-i})$$

- (Pure strategy) **Nash equilibria**: configurations x^* in $\mathcal{X}$ such that

$$x_i^* \in \mathcal{B}_i(x_{-i}^*), \qquad \forall i \in \mathcal{V}.$$

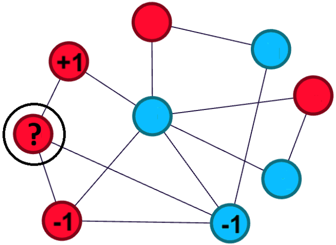
- $\mathcal{N}$ set of Nash equilibria
- We focus on **existence** of Nash equilibria and their **reachability via best response (BR)-paths**:
  - a length-ℓ BR-path from x to y is a sequence $(x^{(0)} = x, \ldots, x^{(\ell)} = y)$ such that $\forall k$, $\exists i_k$ in $\mathcal{V}$ such that

$$x_{-i_k}^{(k)} = x_{-i_k}^{(k-1)}, \qquad x_{i_k}^{(k)} \in \mathcal{B}_{i_k}(x_{-i_k}^{(k-1)}).$$

  - $\mathcal{N}$ is reachable from $x \in \mathcal{X}$ if there is a BR path from x to some $y \in \mathcal{N}$
  - $\mathcal{N}$ is globally reachable if it is reachable from every $x \in \mathcal{X}$

Example

- set of coordinating agents $\mathcal{R}$
- set of anti-coordinating agents $\mathcal{S}$
- No preferences: $d_i = 0$ for all i
- Unweighted graph $\mathcal{G}$ with $W_{ij} \in \{0, 1\}$ for all i, j



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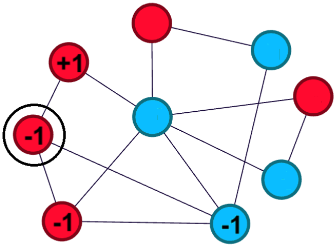

- A coordinating agent "wakes up" and observes the current state
- Then, updates the strategy according to the best-response (coordinates with the majority)

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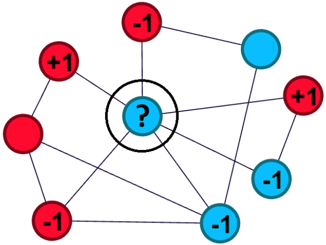

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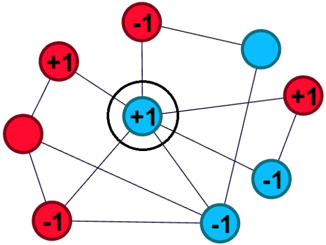

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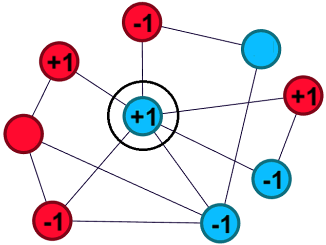

- An anti-coordinating agent "wakes up" and observes the current state
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- Do Nash equilibria exist? Can we "reach" them in finite time?

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Potential games (Monderer and Shapley 1996)

- A game $(\mathcal{V}, \mathcal{A}, \{u_i\}_{i \in \mathcal{V}})$ is an (exact) **potential game** if there exists $\Phi : \mathcal{A}^{\mathcal{V}} \to \mathbb{R}$ (called **potential function**) such that

$$u_i(y_i, x_{-i}) - u_i(x_i, x_{-i}) = \Phi(y_i, x_{-i}) - \Phi(x_i, x_{-i})$$

- For every player, utility variation incurred in changing unilaterally the action is the same as the corresponding variation in potential.
  $\to$ Set of Nash equilibria $\mathcal{N}$ is nonempty and globally reachable
  $\to$ Asynchronous best-response dynamics converges in finite time

Network coordination game ($\mathcal{V} = \mathcal{R}$)

Utilities: $$u_i(x_i, x_{-i}) = \sum_{j \in \mathcal{V}} W_{ij} x_i x_j + d_i x_i$$

- Symmetric two-agent game, undirected graph $\rightarrow$ The network game is potential
- If $d_i = 0$ for all i $\rightarrow$ Potential game (straightforward)
- It actually holds in general

Proposition

If undirected graph, then ***potential function***

$$\Phi_c(x) = \frac{1}{2} \sum_{i,j \in \mathcal{V}} W_{ij} x_i x_j + \sum_{i \in \mathcal{V}} d_i x_i$$

- The set of Nash equilibria is nonempty and globally reachable (Ramazi et al, 2016).
- Characterization of Nash equilibria based on cohesiveness of subsets (Morris, 2000)

Network anti-coordination game ($\mathcal{V} = \mathcal{S}$)

Utilities: $$u_i(x_i, x_{-i}) = -\sum_{j \in \mathcal{V}} W_{ij} x_i x_j - d_i x_i$$

- Symmetric two-agent game, undirected graph $\rightarrow$ The network game is potential
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Proposition

If undirected graph, then ***potential function***

$$\Phi_a(x) = -\Phi_c(x) = -\frac{1}{2} \sum_{i,j \in \mathcal{V}} W_{ij} x_i x_j - \sum_{i \in \mathcal{V}} d_i x_i$$

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- Characterization of Nash equilibria not trivial

Parenthesis: signed networks $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \bar{W})$

Utilities: $u_i(x_i, x_{-i}) = \sum_{j \in \mathcal{V}} \bar{W}_{ij} x_i x_j + d_i x_i$

- $\bar{W}_{ij} \geq 0$ for all $i, j \rightarrow$ network coordination game
- $\bar{W}_{ij} \leq 0$ for all $i, j \rightarrow$ network anti-coordination game
- $\bar{W}_{ij} \in \mathbb{R}$ for all $i, j \rightarrow$ it is still a potential game.

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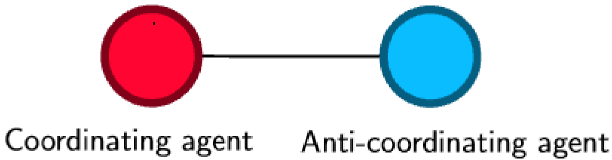
- The set of Nash equilibria is nonempty and globally reachable
- Characterization of Nash equilibria not trivial (related to computation of frustration)

Mixed network coordination/anti-coordination game

Proposition

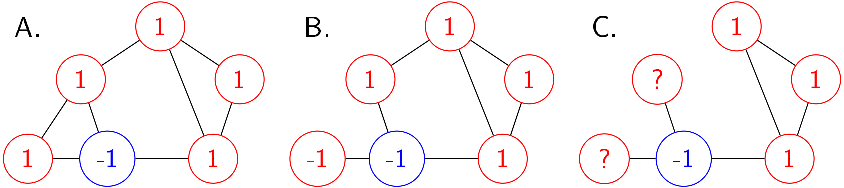
*One **edge** between a **coordinating agent** and an **anti-coordinating agent***
$\rightarrow$ ***not** a potential game*



The discoordination game admits no Nash equilibria

Existence of equilibria depends on network structure

- $\mathcal{R}$ coordinating agents, $\mathcal{S}$ anti-coordinating agents, $d_i = 0$ for all i



- (A) two (symmetric) Nash equilibria where coordinating agents are **at consensus**
- (B) two (symmetric) Nash equilibria where coordinating agents are **not at consensus**
- (C) no Nash equilibria

Convergence to equilibria depends on network structure

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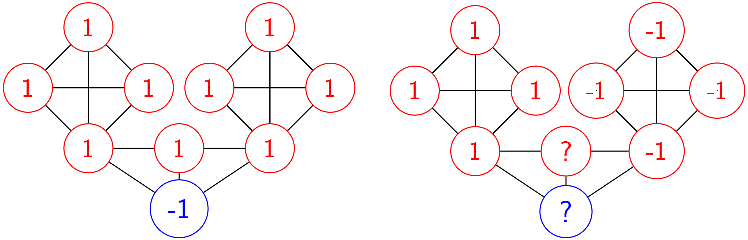

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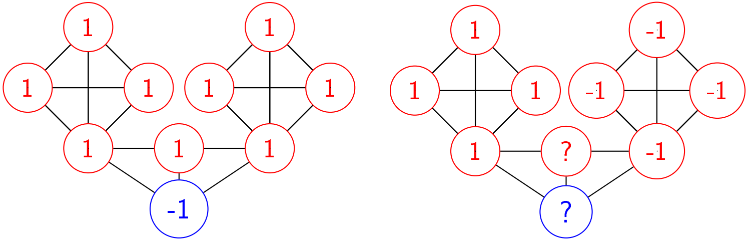

- two (symmetric) Nash equilibria where coordinating agents are **at consensus**
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⋆ even when the set of Nash equilibria is non-empty, it might not be globally reachable

Takeaways

1. Irregular network topology and heterogeneous preferences are not sufficient to cause nonexistence of Nash equilibria; **coexistence of coordinating and anti-coordinating agents** plays a crucial role
2. existence of Nash equilibria depends on the network structure, the value of the threshold parameter and the roles of agents
3. even when the set of Nash equilibria is non-empty, it might not be globally reachable

$\longrightarrow$ The study of Nash equilibria for the mixed network CAC game is a **challenging problem**.

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1. Irregular network topology and heterogeneous preferences are not sufficient to cause nonexistence of Nash equilibria; **coexistence of coordinating and anti-coordinating agents** plays a crucial role
2. existence of Nash equilibria depends on the network structure, the value of the threshold parameter and the roles of agents
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$\longrightarrow$ The study of Nash equilibria for the mixed network CAC game is a **challenging problem**.

We focus on Nash equilibria that are **consensus on the coordinating side**

- how does the presence of anti-coordinating agents and the structure of interconnections affect the behavior of the coordinating agents?

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Preliminary considerations

- Utility of coordinating agent $i \in \mathcal{R}$:

$$u_i(x_i, x_{-i}) = \sum_{j \in \mathcal{V}} W_{ij} x_i x_j + d_i x_i = x_i \left(\sum_{j \in \mathcal{V}} W_{ij} x_j + d_i \right)$$

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Results: existence

- Recall: $\mathcal{B}_i(x_{-i}) = \text{sign}(\sum_{j \text{ playing } 1} W_{ij} - \sum_{j \text{ playing -1}} W_{ij} + d_i)$ for all coordinating agents

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- $\mathcal{R}$ coordinating agents, $\mathcal{S}$ anti-coordinating agents
- Notation: $w_i^{\mathcal{R}} = \sum_{j \in \mathcal{R}} W_{ij}$ and $w_i^{\mathcal{S}} = \sum_{j \in \mathcal{S}} W_{ij}$

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Theorem (Sufficient condition for existence)

If, for all i in $\mathcal{R}$, it holds

$$w_i^{\mathcal{R}} + a d_i \geq w_i^{\mathcal{S}}$$

for some a in $\{\pm 1\}$. Then, the network CAC game admits at least one Nash equilibrium where coordinating agents play action a.

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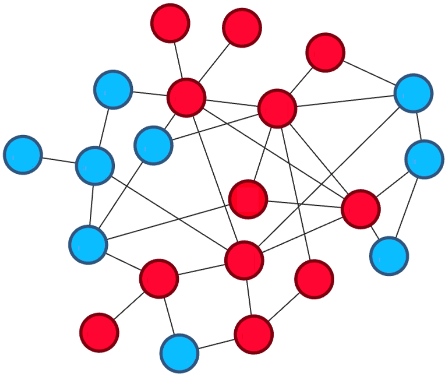
- (*) is a generalization of the idea of **cohesiveness** in (Morris, 2000) (characterization of Nash equilibria in network coordination games).

Example 1: idea behind the proof

- $\mathcal{R}$ coordinating agents, $\mathcal{S}$ anti-coordinating agents, $d_i = 0$ for all i



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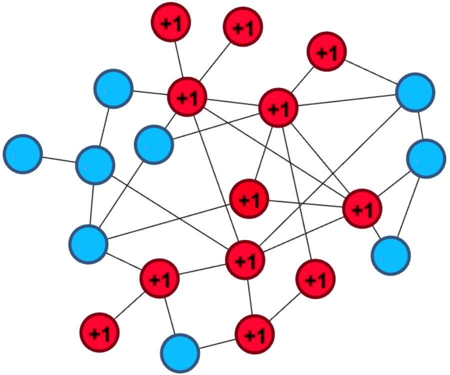

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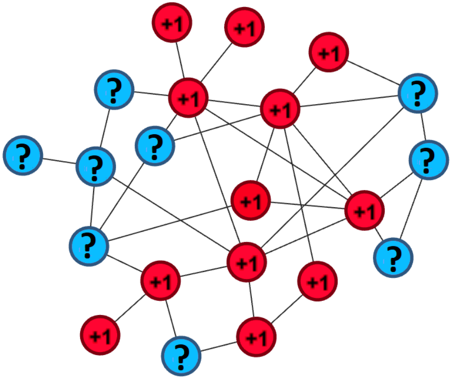

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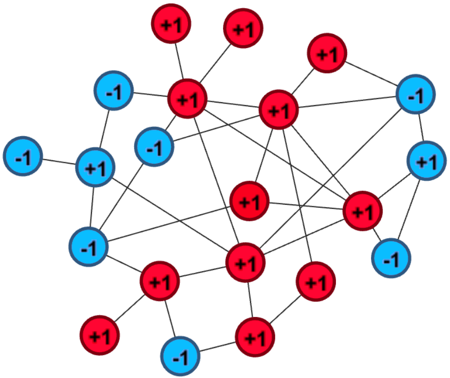

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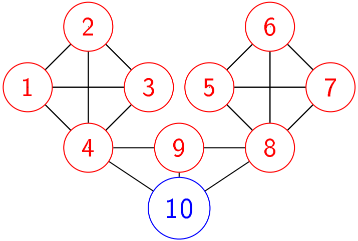

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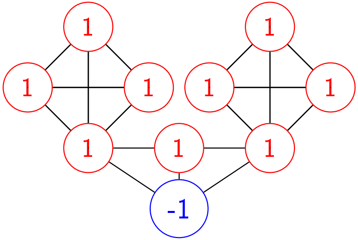

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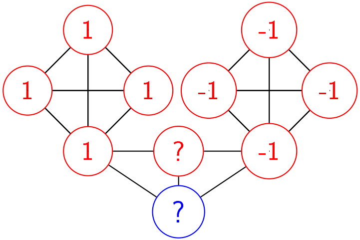

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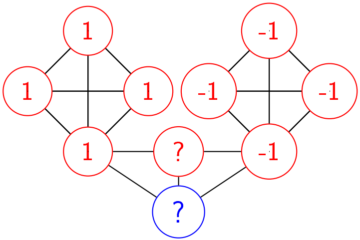

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- Is it globally reachable? No

IDEA: for an action to spread to the whole coordinating population and be stable, there cannot be **sub-groups of coordinating agents** which are **excessively cohesive**

Indecomposability

- External fields h^- and h^+ in $\mathbb{R}^{\mathcal{V}}$ such that $h_i^- \leq h_i^+$ for all i
- $\mathcal{G}_{\mathcal{R}} = (\mathcal{R}, \mathcal{E}, W)$ is (h^-, h^+)**-indecomposable** if for every partition $\mathcal{R} = \mathcal{R}^+ \cup \mathcal{R}^-$, $\exists i \in \mathcal{R}$ such that either

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IDEA: suppose all nodes in $\mathcal{R}^+$ play 1 and all nodes in $\mathcal{R}^-$ play -1. Then, at least one node in $\mathcal{R}^+$ or $\mathcal{R}^-$ is not in equilibrium with the external fiels h_i^+ or h_i^-, respectively.

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$\rightarrow$ cohesiveness is violated by at least one node i in $\mathcal{R}^+$ or in $\mathcal{R}^-$.

Example

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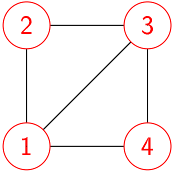

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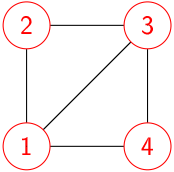

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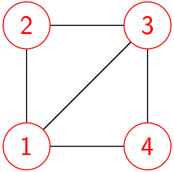

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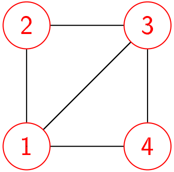

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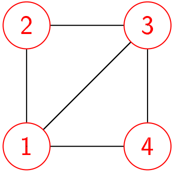

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- ...

General idea:

- node 3 has three edges and no external field $\rightarrow$ it must be in a subset with at least two other nodes
- nodes 2 and 4 have 2 edges and external field between -1 and $+1$ $\rightarrow$ cannot be alone.
- nodes 1 has 3 edges and external field between -2 and $+2$ $\rightarrow$ cannot be alone.

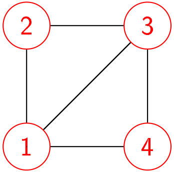
Example

- $\mathcal{G}_{\mathcal{R}}$ is (h^{-}, h^{+})-**indecomposable** if for every partition $\mathcal{R} = \mathcal{R}^{+} \cup \mathcal{R}^{-}$, $\exists i \in \mathcal{R}$ such that either

$$i \in \mathcal{R}^{+} \text{ and } w_i^{\mathcal{R}^{+}} + h_i^{+} < w_i^{\mathcal{R}^{-}} \quad \text{or} \quad i \in \mathcal{R}^{-} \text{ and } w_i^{\mathcal{R}^{-}} - h_i^{-} < w_i^{\mathcal{R}^{+}}.$$

Example:

- $\mathcal{R}$ is **not** (h^{-}, h^{+})-indecomposable for $h^{-} = -\mathbb{1}$ and $h^{+} = \mathbb{1}$



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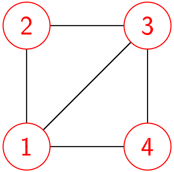

- $\mathcal{R}$ is **not** (h^-, h^+)-indecomposable for $h^- = -\mathbb{1}$ and $h^+ = \mathbb{1}$
- $\mathcal{R}^+ = \{1, 2\}$ and $\mathcal{R}^- = \{3, 4\}$
- $w_1^{\mathcal{R}^+} + h_1^+ = 2 \geq 2 = w_1^{\mathcal{R}^-}$
- $w_2^{\mathcal{R}^+} + h_2^+ = 2 \geq 1 = w_2^{\mathcal{R}^-}$
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Results: reachability

- $\mathcal{G}_{\mathcal{R}}$: graph restricted to the set of coordinating agents $\mathcal{R}$
- $\mathcal{G}_{\mathcal{R}}$ is (h^-, h^+)-**indecomposable** if for every partition $\mathcal{R} = \mathcal{R}^+ \cup \mathcal{R}^-$, $\exists i \in \mathcal{R}$ such that either

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- **IDEA**:
  - max external field: $h_i^+ = d_i + w_i^{\mathcal{S}}$ ("worst case scenario": anti-coordinating agents play 1)
  - min external field: $h_i^- = d_i - w_i^{\mathcal{S}}$ ("worst case scenario": anti-coordinating agents play -1)

Theorem (Sufficient condition for stability)

Assume that

- *for all i in $\mathcal{R}$, $w_i^{\mathcal{R}} + ad_i \geq w_i^{\mathcal{S}}$ for some a in $\{\pm 1\}$ (**cohesiveness**)*
- *$\mathcal{G}_{\mathcal{R}}$ is (h^-, h^+)-**indecomposable** in $\mathcal{G}_{\mathcal{R}}$ with $h_i^+ = d_i + w_i^{\mathcal{S}}$ and $h_i^- = d_i - w_i^{\mathcal{S}}$ for all i*

Then the set of Nash equilibria where coordinating agents are at consensus is non-empty and globally reachable.

Example

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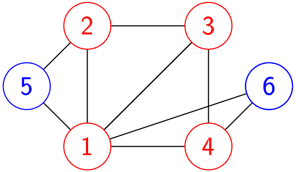
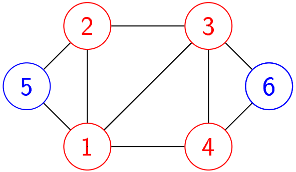


- $w_i^{\mathcal{R}} \geq w_i^{\mathcal{S}}$ for all i in $\mathcal{V}$ $\rightarrow$ existence of NE
- $w^{\mathcal{S}} = (2, 1, 0, 1) \rightarrow h^+ = (2, 1, 0, 1)$ and $h^- = -h^+$
- $\mathcal{G}_{\mathcal{R}}$ is (h^-, h^+)-indecomposable
  $\rightarrow$ the set of NE where coordinating agents are at consensus is nonempty and globally reachable



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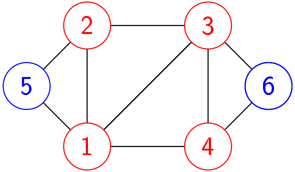

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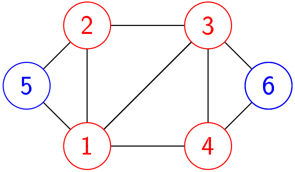

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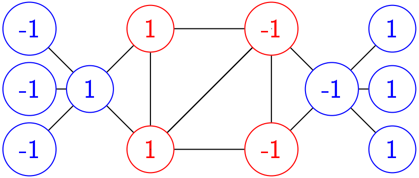
Example of Nash equilibrium where coordinating agents are **not** at consensus:

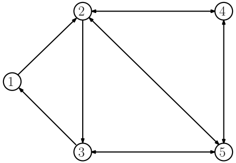


Extensions: Linear Threshold Dynamics with external field $h(t)$

- **Directed** graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$
- state $X_i(t)$ in $\{-1, 1\}$ for all i in $\mathcal{V}$
- (Independent rate-1) Poisson clock of agent i clicks at time $t > 0$:

$$X_i(t) = \begin{cases} +1 & \text{if} \quad \sum_j W_{ij} X_j(t) + h_i(t) > 0 \\ X_i(t^-) & \text{if} \quad \sum_j W_{ij} X_j(t) + h_i(t) = 0 \\ -1 & \text{if} \quad \sum_j W_{ij} X_j(t) + h_i(t) < 0 \,. \end{cases}$$

- Necessary and sufficient conditions for global stability of consensus equilibria



Linear Threshold Dynamics with external field $h(t)$

- Directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$
- $h^- \leq h(t) \leq h^+$
- $\mathcal{G}$ **not** (h^-, h^+)-indecomposable
- $\exists h^* \in (h^-, h^+)$ such that it polarizes with $h(t) = h^*$ for all t

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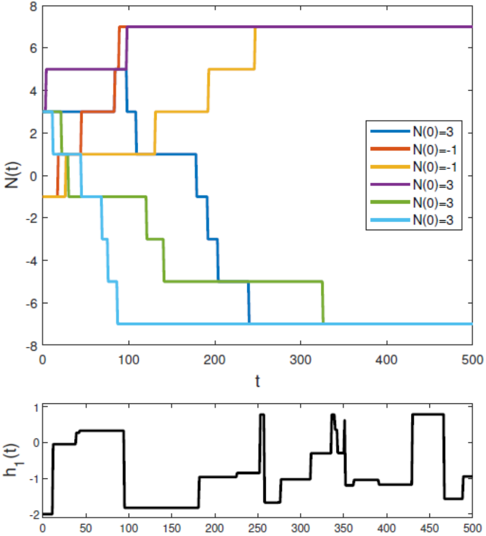

- $\mathcal{G}$ is (h^-, h^+)-indecomposable

⋆ CASE 1: $w_i \geq -h_i^-$ and $w_i \geq h_i^+$ for all i

⋆ absorbed in finite time in a consensus configuration ($N(t) = \sum_i X_i(t)$)

Linear Threshold Dynamics with external field $h(t)$

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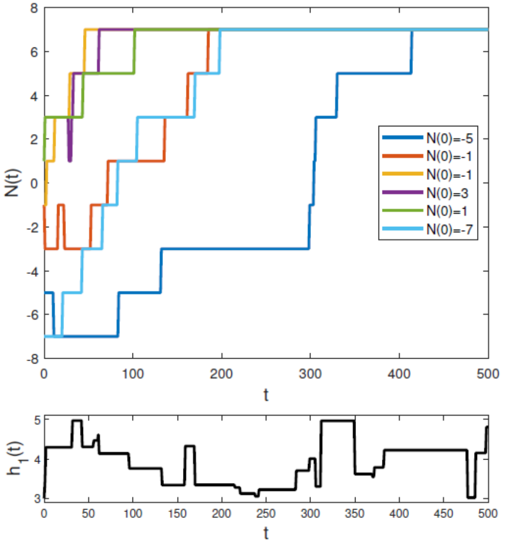
⋆ CASE 2: $w_i \geq -h_i^-$ and $w_i \not\geq h_i^+$ for all i

⋆ absorbed in finite time in $x^* = +1$ $(N(t) = \sum_i X_i(t))$



Linear Threshold Dynamics with external field $h(t)$

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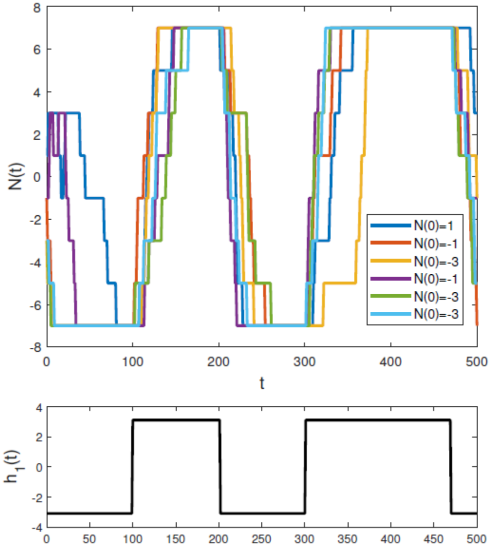
- $h^- \leq h(t) \leq h^+$
- $\mathcal{G}$ is (h^-, h^+)-indecomposable

⋆ CASE 3: $w_i \ngeq -h_i^-$ and $w_i \ngeq h_i^+$ for all i

⋆ there exists $h(t)$ such that $X(t)$ fluctuates forever $(N(t) = \sum_i X_i(t))$



Outline

1. Motivation and problem description
2. Related work and main issues
3. Results
4. Conclusions and extensions

Summary

- Network games with a mixture of coordinating and anti-coordinating agents
- Sufficient conditions on **network topology** for **existence and reachability** of Nash equilibria that are **consensus** on the side of coordinating agents [1]

⋆ generalized role of cohesiveness in mixed games

⋆ novel notion of indecomposability

Summary

- Network games with a mixture of coordinating and anti-coordinating agents
- Sufficient conditions on **network topology** for **existence and reachability** of Nash equilibria that are **consensus** on the side of coordinating agents [1]

⋆ generalized role of cohesiveness in mixed games

⋆ novel notion of indecomposability

Related work[2]

- complete characterization of the set of Nash equilibria in the **complete** graph (no network structure, conditions on thresholds)

[1] Arditti, Como, Fagnani, V., (CDC 2021) [2] V., Como, Fagnani, Arditti (IFAC 2020)

Extensions

Interpretation of the results:

- **robustness** of pure network coordination games against the change of behavior of a subset of agents

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Linear Threshold Dynamics (LTD) with **time-varying** threshold rule (exogenous signal, external influence) on weighted **directed** interaction networks [3]

- Necessary and sufficient conditions for global stability of consensus equilibria
- Novel notion of robust improvement paths

⋆ pure coordinating games are supermodular (best response dynamics coincide with LTD)

[3] Arditti, Como, Fagnani, V., *Robust Coordination of Linear Threshold Dynamics on Directed Weighted Networks* (Submitted to IEEE TAC)

Current and further work

Current work:

- Mixed network CAC games on **directed graphs**
- Supermodular property of network coordination games

Further work

- Extensions and connections to signed graphs
- Necessary conditions
- Nonconsensus equilibria
- Non independent external fields
- Computational tractability of indecomposability

Thank you for your attention!