Robust inverse optimal control

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Motivation: Direct Optimal Control

Continuous-time infinite horizon

$$V(x_0) := \min_{u \in \mathbb{R}^m} \max_{w \in \mathbb{R}^n} \int_0^\infty q(x(s)) + u^\top(s)\, R\, u(s) - \xi\, w^\top(s)\, S\, w(s) \mathrm{d}s$$

$$\text{subject to } \dot{x} = f(x) + G_1^\top(x) u + G_2^\top(x) w,$$

$$x(0) = x_0$$

- $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ is continuous, locally Lipschitz and $f(0) = 0$
- $G_k(x) = [g_1^\top(x), \ldots, g_m^\top(x)]^\top$ with $k = 1, 2$ and $g_i : \mathbb{R}^n \mapsto \mathbb{R}^n, i = 1 \ldots m$ is continuous
- $R = R^\top$, $S = S^\top > 0, \xi > 0$ robustness to unmodelled dynamics
- $q : \mathbb{R}^n \mapsto \mathbb{R}_+$ is continuous and $q(0) = 0$

Find the optimal value $V(x_0)$ associated with an optimal controller $u^*(x) := \arg\min_{u \in \mathbb{R}^m} V(x_0)$

Motivation: Direct optimal control

1 **opt. problem**

$$\min_u \max_w \int_0^\infty [q(x) + u^\top R u - \xi w^\top S w]\, ds$$

$$\text{s.t. } \dot{x} = f(x) + G^\top u + \overline{G}^\top w,$$

$$x(0) = x_0$$

2 **HJI equations**

$$V(x_0) := \inf_u \sup_w \int_0^\infty [q(x) + u^\top R u - \xi w^\top S w]\, ds$$

$$\min_u \max_w \{q(x) + u^\top R u - \xi w^\top S w + \nabla_x^\top V \dot{x}\} = 0$$

3 **optimal controller**

$$u^*(x) = -\frac{1}{2} R^{-1} G \nabla_x V$$

3 **optimal controller w.r.t.**

$$\min_u \max_w \int_0^\infty [q(x) + u^\top R u - \xi w^\top S w]\, ds$$

$$\text{s.t. } \dot{x} = f(x) + G^\top u + \overline{G}^\top w,$$

$$x(0) = x_0$$

2 **cost design from HJI**

$$q(x) = -\nabla_x V^\top (f + G^\top u^* + \overline{G}^\top w^*) - |u^*|_R^2 + \xi |w^*|_S^2$$

$$\min_u \max_w \{q(x) + u^\top R u - \xi w^\top S w + \nabla_x^\top V \dot{x}\} = 0$$

1 **CLF and controller**

$$\nabla_x V^\top \left(f + G^\top u^* + \overline{G}^\top w^*\right) < -u^{*\top} R u^* + \xi w^{*\top} S w^*$$

$$u^*(x) = -\frac{1}{2} R^{-1} G \nabla_x V$$

$$\dot{x} = f(x) + G^\top u + \overline{G}^\top w, \quad x(0) = x_0$$

Inverse optimal control: Problem setup

Ingredients for the optimal control problem:

- Input-affine nonlinear system dynamics - unknown disturbances

$$\dot{x} = f(x) + G_1^\top(x)u + G_2^\top(x)w$$

- Unknown performance criteria - unknown positive definite $q(x)$,

$$\min_{u\in\mathbb{R}^m} \max_{w\in\mathbb{R}^n} \int_0^\infty q(x(s)) + u^\top(s)\,R\,u(s) - \xi\,w^\top(s)\,S\,w(s)\mathrm{d}s$$

1. How to exploit cost design for (robust) optimal control to circumvent numerically solving PDEs?
2. How to derive optimal feedback control laws in networks that inherit topological structure?

Inverse optimal control: Problem setup

Given is the feedback stabilizing controller

$$u^*(x) = -\frac{1}{2} R^{-1} G_1(x) \nabla_x V \tag{1}$$

with a control Lyapunov function $V : \mathbb{R}^n \mapsto \mathbb{R}_+$ and $V(x) > 0$, $V(0) = 0$,

$$\nabla_x^\top V\big(f(x) + G_1^\top(x) u^*(x) + G_2^\top w^*(x)\big) < -u^{*\top}(x)\, R\, u^*(x) + \xi\, w^{*\top}(x) S w^*(x)$$

with $w^*(x) = -\frac{1}{2\xi} S^{-1} G_2(x) \nabla_x V$ and consider

$$\begin{aligned} \min_{u \in \mathbb{R}^m} \max_{w \in \mathbb{R}^n} & \int_0^\infty q(x(s)) + u^\top(s)\, R\, u(s) - \xi\, w^\top(s)\, S\, w(s) \mathrm{d}s \qquad (2) \\ \text{subject to } & \dot{x} = f(x) + G_1^\top(x) u + G_2^\top(x) w, \\ & x(0) = x_0 \end{aligned}$$

Determine the function $q(x)$ so that (2) has the optimal value $V(x_0)$ and the control solution u^* in (1)

Robust inverse optimal control: Main result

Consider the robust optimal control problem (2) *together with the control Lyapunov function V associated with the controller u^* in* (1) *and the worst-case disturbance $w^*(x) = -\frac{1}{2\xi} S^{-1} G_2(x) \nabla_x V$. Assume that*

$$\nabla_x^\top V \left(f(x) + G_1^\top(x) u^*(x) + G_2^\top(x) w^*(x) \right) < - u^{*\top}(x) \, R \, u^*(x) + \xi \, w^{*\top}(x) S \, w^*(x),$$

and

$$q(x) = -\nabla_x^\top V \left(f(x) + G_1^\top(x) u^*(x) + G_2^\top(x) w^*(x) \right) - u^{*\top}(x) \, R \, u^*(x) + \xi \, w^{*\top}(x) S \, w^*(x).$$

Then, the robust optimal control problem (2) *has the optimal value $V(x_0)$ and the optimal control u^*.*

Proof idea

1 opt. problem

$$\min_u \max_w \int_0^\infty [q(x) + u^\top R u - \xi w^\top S w]\, ds$$

$$\text{s.t. } \dot{x} = f(x) + G^\top u + \overline{G}^\top w, \quad x(0) = x_0$$

2 HJI equations

$$V(x_0) := \inf_u \sup_w \int_0^\infty [q(x) + u^\top R u - \xi w^\top S w]\, ds$$

$$\min_u \max_w \{q(x) + u^\top R u - \xi w^\top S w + \nabla_x^\top V \dot{x}\} = 0$$

3 optimal controller

$$u^*(x) = -\frac{1}{2} R^{-1} G \nabla_x V$$

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3 optimal controller w.r.t.

$$\min_u \max_w \int_0^\infty [q(x) + u^\top R u - \xi w^\top S w]\, ds$$

$$\text{s.t. } \dot{x} = f(x) + G^\top u + \overline{G}^\top w, \quad x(0) = x_0$$

2 cost design from HJI

$$q(x) = -\nabla_x V^\top (f + G^\top u^* + \overline{G}^\top w^*) - |u^*|_R^2 + \xi |w^*|_S^2$$

$$\min_u \max_w \{q(x) + u^\top R u - \xi w^\top S w + \nabla_x^\top V \dot{x}\} = 0$$

1 CLF and controller

$$\nabla_x V^\top \left(f + G^\top u^* + \overline{G}^\top w^*\right) < -u^{*\top} R u^* + \xi w^{*\top} S w^*$$

$$u^*(x) = -\frac{1}{2} R^{-1} G \nabla_x V$$

$$\dot{x} = f(x) + G^\top u + \bar{G}^\top w, \quad x(0) = x_0$$

Direct vs. inverse optimal control

Direct optimal control

1. No explicit analytical solution in general
2. HJB or HJI (ctn.)/ Bellman (disc.) eq. hard to solve
3. Challenging cost design

Inverse optimal control

1. No numerical effort involved
2. Classical stabilizing control/ CLF methods
3. Intuitive control tuning

Discussion

- Associating a criterion (i.e., via the cost to minimize) to a stabilizing controller (among possibly many) is useful
- Robustness guarantees: gain margin of $[1/2, \infty)$ for a diagonal input matrix $R = R^\top > 0$, analog to LQR
- Tuning aspect: solve a family of optimization problems with the same value function V
  - The matrices S and R can be tuned for $R' \leq R$ and $S' \geq S$
  - If $\nabla_x^\top V\big(f(x) + G_2^\top(x) w^*(x)\big) < 0$, the origin is asymptotically stable under the action of the worst case disturbance w^* and R, S can be tuned arbitrarily

$\Rightarrow$ Minimize control effort and improve error decay rate

Example 1: Linear systems

Consider the linear system

$$\dot{x} = Ax + Bu + \overline{B}w, \quad x(0) = x_0$$

where $\overline{B} \in \mathbb{R}^{n \times n_w}$ is disturbance input matrix and $w \in \mathbb{R}^{n_w}$ is unknown additive disturbance. The stabilizing controller is given by,

$$u^*(x) = -\frac{1}{2}R^{-1}B^\top P x.$$

with $V(x) = \frac{1}{2}x^\top Px$, $P = P^\top > 0$. We define the cost function,

$$L(x, u, w) = x^\top \overbrace{Q(R, S)}^{M^\top M} x + u^\top Ru - \xi w^\top Sw, \ \xi > 0.$$

For (A, B) controllable and (A, M) observable, let $P, R, S > 0$ satisfy

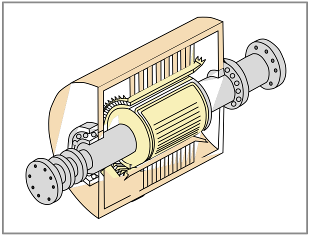
$$\frac{1}{4}PBR^{-1}B^\top P - \frac{1}{4\xi}P\overline{B}S^{-1}\overline{B}^\top P - \frac{1}{2}(PA + A^\top P) > 0$$

then, $Q(R, S) = \frac{1}{4}PBR^{-1}B^\top P - \frac{1}{4\xi}P\overline{B}S^{-1}\overline{B}^\top P - \frac{1}{2}(PA + A^\top P)$

Example 2: Inverter-based power generation

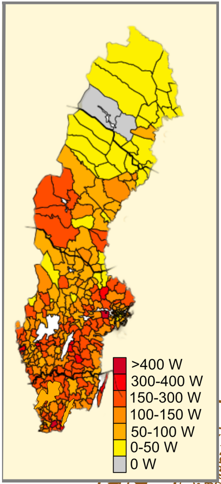
- Optimization for stability and control in power systems
- A paradigm shift in power generation
- Transition on the device- and system-level
- From synchronous machines to inverters

⇒ Rethink of desired grid operation



Example 2: Control of inverter networks

How to leverage IOC theory and available power measurements to design optimal (locally) stabilizing and implementable (a.k.a feasible) controllers?



Example 2: Angular droop control

- A network of $n-$ inverters with m inductive lines
- Constant voltage magnitude and quasi steady state
- Given nominal phase angles $\theta^*(t) = \omega_0\, t + \theta_0^*$
- Let $\theta^s = \{\theta_i^s\}_{k=1}^n$ satisfying $\gamma(\theta^s - \theta^*) + P_e(\theta^s) - P_e^* = 0$
- Controllable phase angle dynamics described by an integrator

$$
\begin{aligned}
\underset{u \in \mathbb{R}^n}{\text{minimize}} \quad & \int_0^\infty \left(\sum_{i=1}^n q_i(\theta) + \alpha_i u_i^2(\theta) \right) \mathrm{d}\tau \\
\text{subject to} \quad & \dot{\theta} = u(\theta) + \omega_0, \\
& \theta(0) = \theta_0 .
\end{aligned}
$$

$q_i(\theta)$ is positive definite w.r.t. θ^s and unknown.

Example 2: Angular droop control

- Security constraint $(\star): \mathscr{B}^\top \theta^s \in (-\frac{\pi}{2}, \frac{\pi}{2})^m$, where $\mathscr{B}$ is the graph incidence matrix
- Under security constraint $(\star)$, the controller

$$\boxed{u_i^*(\theta) = -\frac{1}{2\alpha_i}\left(\gamma_i(\theta_i - \theta_i^*) + P_{e,i}(\theta) - P_{e,i}^*\right),\ i = 1, \ldots, n}$$

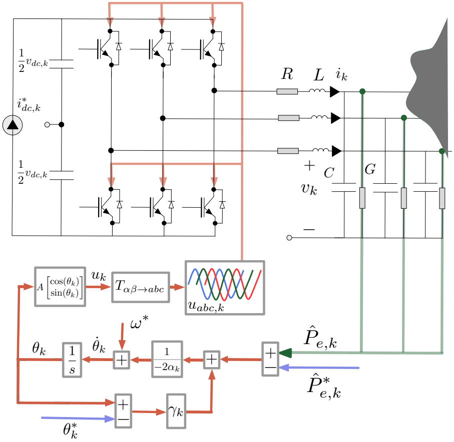
is locally stabilizing for the angle dynamics with $u_i^*(\theta^s) = 0$ and

$$V(\theta) = \frac{1}{2}|(\theta - \theta^s|_Z^2 + \sum_{i=1}^{n} \sum_{j \in \mathcal{N}_i} b_{ij}(\cos(\theta_{ij}) - \cos(\theta_{ij}^s)) - (\theta_{ij}^0 - \theta_{ij}^s)\sin(\theta_{ij}^s)$$

- The controller u_i^* is locally optimal w.r.t.

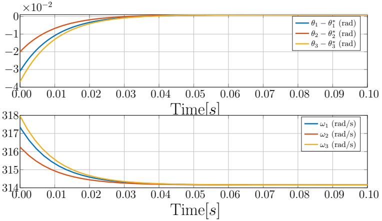
$$\int_0^\infty \left(\sum_{k=1}^{n} \left[\alpha_i u_i^2(\theta) + \underbrace{\frac{1}{4\alpha_i}(\gamma_i(\theta_i - \theta_i^*) + P_{e,i}(\theta) - P_{e,i}^*)}_{q_i(\theta)}\right]\right) d\tau$$

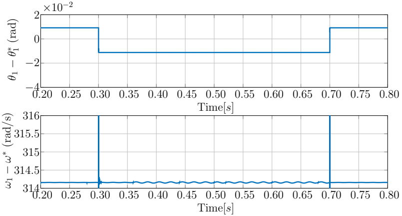
Example 2: Feasible implementation



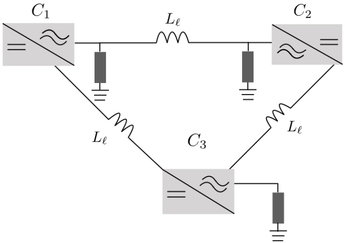
grid-forming and fully decentralized controller

Example 2: Properties of the A-droop control





- Zero frequency error
- Angle-to-power droop
- Intuitive control tuning (LQR)
- Scalability to large networks



Example 3: Angle control

$$
\begin{aligned}
\min_u \max_w \int_0^\infty & q(\delta(s), \omega(s)) + |u(s)|_R^2 - \xi\, |w(s)|_S^2 \,\mathrm{d}s \\
\text{s.t.} \quad & \dot{\delta} = \mathscr{B}^\top \omega + u, \\
& M\dot{\omega} = -D\,\omega - \mathscr{B}\, L \left(\underline{\sin}(\delta) - \underline{\sin}(\delta^*) \right) + w, \\
& (\delta(0), \omega(0)) = (\delta_0, \omega_0),
\end{aligned} \tag{4}
$$

- $\mathscr{B}$ is the graph incidence matrix
- $\delta = \mathscr{B}^\top \theta \in \mathbb{R}^m$ neighboring angle difference vector
- $M, D, L > 0$ (inertia, damping and line susceptance) diagonal matrices
- $w \in \mathbb{R}^n$ power disturbance

Example 3: Angle control

For $\delta^s \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cap \mathrm{Im}(B^\top)$

$$0 = \mathscr{B} L \left(\underline{\sin}(\delta^s) - \underline{\sin}(\delta^*)\right) \tag{5}$$

and $D - \frac{1}{4\xi} S^{-1} > 0$, the controller

$$\boxed{u^*(\delta) = -\frac{1}{2} R^{-1} L (\underline{\sin}(\delta) - \underline{\sin}(\delta^s))}$$

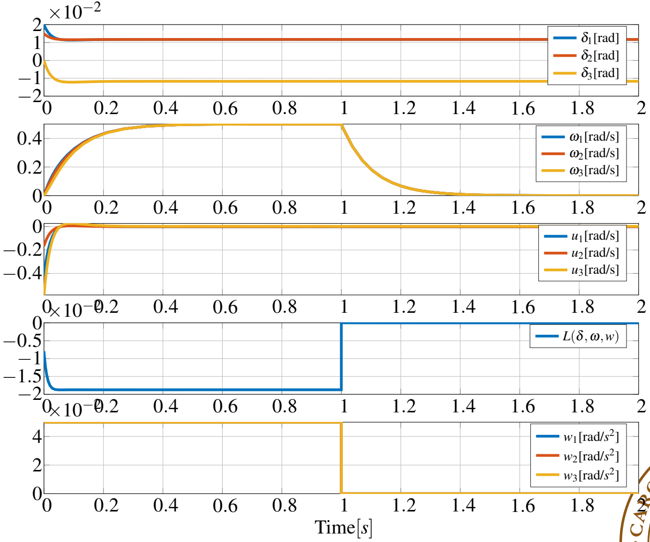
is locally (in a neighborhood of δ^s) optimal w.r.t. (4)

$$q(\delta, \omega) = \frac{1}{4} |\underline{\sin}(\delta) - \underline{\sin}(\delta^s)|^2_{L R^{-1} L} + |\omega|^2_{D - \frac{1}{4\xi} S^{-1}},$$

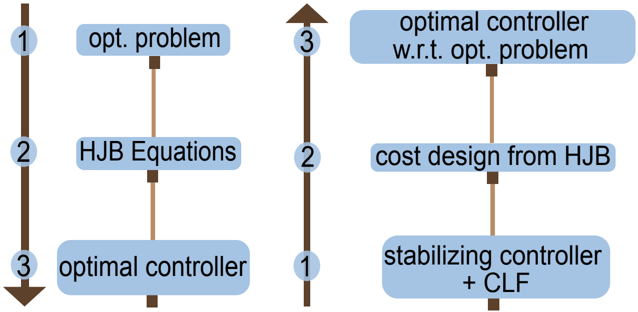
with the value function

$$V(\delta, \omega) = \frac{1}{2} |\omega - \omega^s|^2_M - \mathbb{1}_n^\top L (\underline{\cos}(\delta) - \underline{\cos}(\delta^s)) - (\delta - \delta^s)^\top L \underline{\sin}(\delta^s). \tag{6}$$

Example 3: Angle control



Takeaways



1. Robust inverse optimality for stabilizing feedback controllers
2. Feasible control implementation in (power) networks