Concentration Inequalities in Gossip Opinion Dynamics over Random Graphs

Yu Xing and **Karl H. Johansson**

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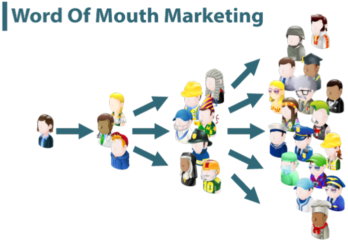
Yu Xing & Karl H. Johansson, "Transient behavior of gossip opinion dynamics with community structure," *arXiv preprint arXiv:2205.14784*, 2022.

———, "Concentration in gossip opinion dynamics over random graphs," *arXiv preprint arXiv:2301.05352*, 2023.

———, "Almost exact recovery in gossip opinion dynamics over stochastic block models," *IEEE Conference on Decision and Control*, 2023.

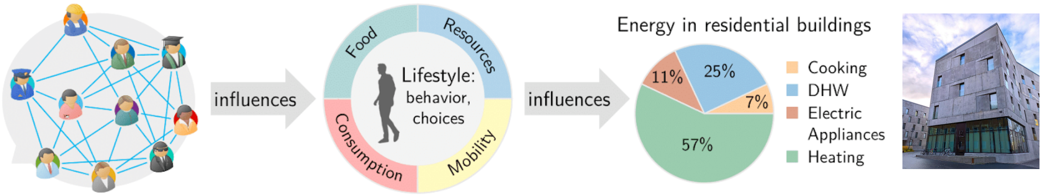
Motivation: Understanding Opinion Dynamics



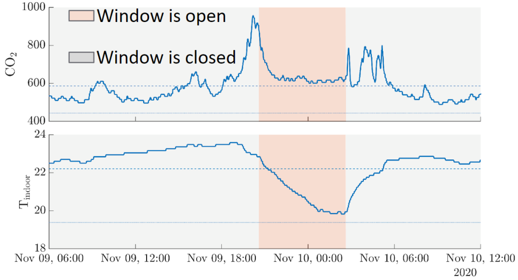
www.science.org/toc/science/381/6656

www.sydsvenskan.se/2020-05-27/skulpturer-i-lund-vill-inte-sprida-coronavirus

Motivation: Social Interactions Influence Building Energy Effieincy

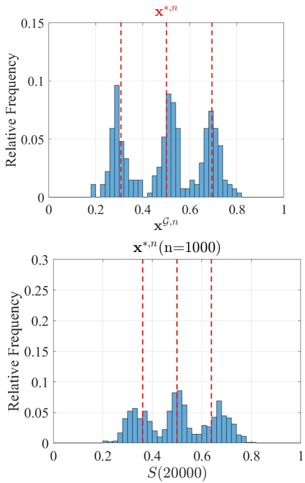


- Smart building sustainability depends on tenants' lifestyle and behavior
- Individuals may change their behavior depending on social interactions [Fontan et al., IFAC CPHS 2022; IFAC WC 2023]
- Experimental study at KTH Live-In-Lab with 250 student residents [Farjadnia et al., IEEE CCTA 2023]

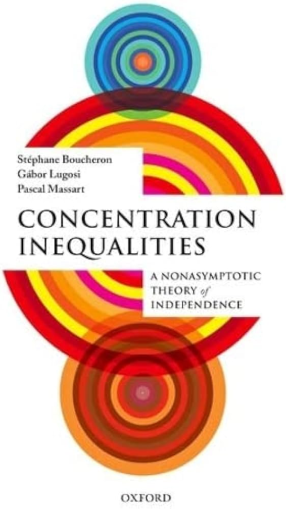


Main Contribution

- **Traditionally,** opinion dynamics research has focused on asymptotic behavior (convergence,...) and qualitative characterization of distributions (consensus, polarization, ...).
- **In this talk,** results are presented on **transient behavior** of gossip opinion dynamics and **quantitative characterization** of opinion distributions.
- We leverage **concentration inequalities** to establish **high-probability bounds over transient time and limited network size.**



Concentration Inequalities



- **Markov's inequality**: $\mathbb{P}\{X \geq a\} \leq \dfrac{\mathbb{E}\{X\}}{a}$.
- **Chernoff's inequality**: $\mathbb{P}\{S_N \geq a\} \leq e^{-\mu}\left(\dfrac{e\mu}{a}\right)^a$.
- **Bernstein's inequality**: $\mathbb{P}\left\{\left\| \sum_{i=1}^{N} Y_i \right\| \geq a\right\} \leq 2n \exp\left(-\dfrac{t^2/2}{\sigma^2 + Kt/3}\right)$.
- **Adjacency matrix of a random graph**: $\mathbb{P}\{\|A - \mathbb{E}\{A\}\| \leq c_1\sqrt{np}\} \geq 1 - n^{-c_2}$.

We derive concentration inequalities for gossip opinion dynamics over random graphs.

Opinion dynamics: individual opinions evolve through social interactions.

Continuous-State Models:

French–DeGroot [French, '56; DeGroot, '74]
Friedkin–Johnsen [Friedkin & Johnsen, '90]
Bounded confidence [Hegselmann&Krause, '02; Deffuant, '00]
Biased assimilation [Dandekar, '13]
Signed Networks [Altafini, '13; Shi, '19]
...

Discrete-State Models:

Voter [Holley & Liggett, '75]
Threshold model [Granovetter, '78]
Majority rule [Galam, '02]
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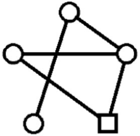
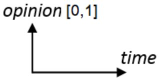
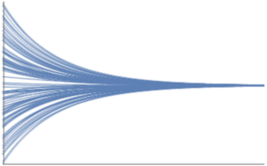
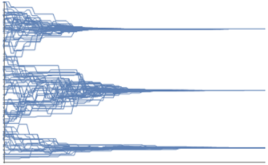
- Graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, with $\mathcal{V} = \{1, \ldots, n\}$ the agent set, $\mathcal{E}$ the edge set, A the weighted adjacency matrix
- Agent i has opinion $X_i(t) \in \mathbb{R}$ at time t



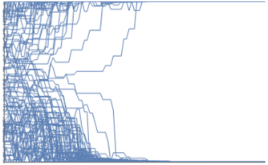




Consensus (DeGroot)



Clustering (Bounded confidence)



Polarization (Signed network)

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Eulerian Approach/Mean-Field Approximation:

[Como & Fagnani, '11; Canuto, '12; Mirtabatabaei, '14; Kolarijani, '21; Ravazzi, '23]

Mean-Field Games over Graphons:

[Caines & Huang, '21; Bayraktara & Wu, '22; Parise & Ozdaglar, '23]

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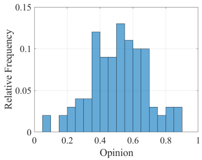
- Agent i has opinion $X_i(t) \in \mathbb{R}$ at time t
- Consider the empirical measure $\mu_t = \frac{1}{n} \sum_i \delta_{X_i(t)}$ and its evolution (where $n \to \infty$), instead of evolution of single agents

$$\begin{bmatrix} X_1(t) \\ \vdots \\ X_n(t) \end{bmatrix} \Rightarrow$$



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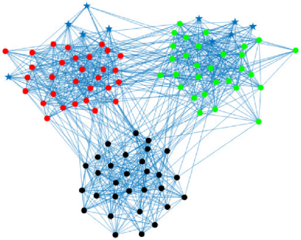
[Caines & Huang, '21; Bayraktara & Wu, '22; Parise & Ozdaglar, '23]

Co-Evolution: [Zino, '20; Fontan, '22]
Multidimensional Opinions:
[Friedkin, '16; Parsegov, '17]
Issue Sequences: [Tian, '18; Wang, '22]
Continuous Opinions and Discrete Actions: [Martins, '08]
...

Network structure highly affects evolution of opinion dynamics.

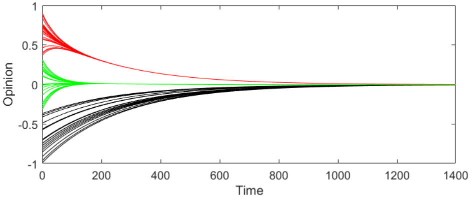


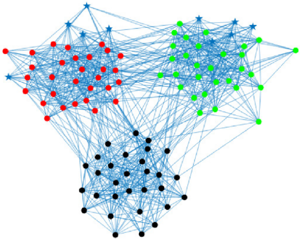
Communities (modules, clusters) are subgroups with dense links internally or similar features/roles, and can be defined by

- partitions based on optimizing modularity
- subgroups controlling the probability of edge existence, e.g., in stochastic block models

Network structure highly affects evolution of opinion dynamics.

- **Influence of community structure on opinion evolution and decision making outcomes**: [Si, '09; Gargiulo, '10; Schaub, '16; Oestereich, '19; Fennell, '21; Peng, '22; Leng, '23]
- **Community detection algorithms based on designed dynamics over known networks**:
  Generic framework: [Schaub, '19]
  Random walks (Infomap, etc): [Rosvall, '08; Delvenne, '10; Lambiotte, '14]
  Bounded confidence model: [Morarescu, '10]
- **Fast and slow dynamics over clustered networks**: [Chow & Kokotovic, '85; Yu, '20; Dutta, '22; Adhikari, '22]





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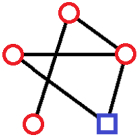
Gossip Dynamics with Stubborn Agents

- Undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ without self-loops.
- $\mathcal{V}$ consists of **regular** agents $\mathcal{V}_{\mathrm{r}}$ and **stubborn** agents $\mathcal{V}_{\mathrm{s}}$. The stubborn agents do not change their opinions (leaders, political parties, or media sources attempting to influence public opinions).



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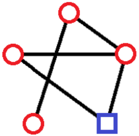

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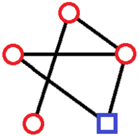

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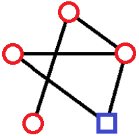

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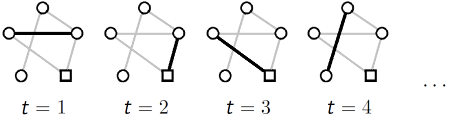
- **Gossip interaction:** Interaction probability matrix $W = [w_{ij}] = A/|\mathcal{E}|$. At time $t \in \mathbb{N}$, with probability w_{ij} independently of previous selections, two agents i and j are selected to interact.

... it is obvious that interpersonal influences do not occur in the simultaneous way ...

— *Friedkin & Johnsen, 1999*



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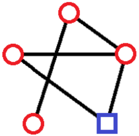

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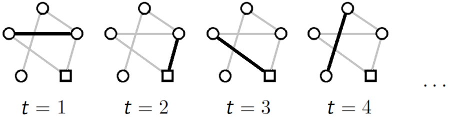

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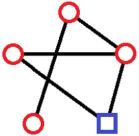

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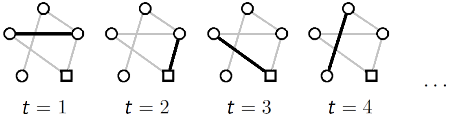

- **Update rule:** Only regular agents of i and j update. If i is regular,

$$X_i(t+1) = \begin{cases} \frac{1}{2}(X_i(t) + X_j(t)), & \text{if } j \text{ is regular,} \\ \frac{1}{2}(X_i(t) + z_j^{(\mathsf{s})}), & \text{if } j \text{ is stubborn.} \end{cases}$$

  If j is regular, it follows the same equation but with i and j swapped.

Random Graph Model

Definition

- Network size: $n \in \mathbb{N}_+$.
- Link probability matrix: $\Psi = \Psi^\mathsf{T} = [\psi_{ij}] \in [0,1]^{n \times n}$.

In the **random graph model** $\mathrm{RG}(n, \Psi)$, a random graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ is constructed by adding undirected edge $\{i, j\}$ to $\mathcal{E}$ with probability ψ_{ij} independent of other agent pairs, where $i \neq j \in \mathcal{V} = \{1, \ldots, n\}$.

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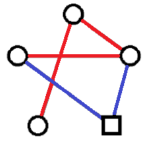
Examples

(i) If $\psi_{ij} \equiv \psi \in [0,1]$, $\forall i, j$, $\mathrm{RG}(n, \Psi)$ is the Erdős–Rényi model.

(ii) Let $\psi_{ij} = w_i w_j / (\sum_k w_k)$, where $w_i \geq 0$ and $\max_i w_i^2 < \sum_k w_k$. Then in $\mathrm{RG}(n, \Psi)$, random graphs have expected fixed degree distribution $\{w_i\}$.

Random Graph Model with Stubborn Agents

Definition



- Agent set: $\mathcal{V} = \mathcal{V}_\mathrm{r} \cup \mathcal{V}_\mathrm{s}$, $\mathcal{V}_\mathrm{r} = \{1, \ldots, n_\mathrm{r}\}$, $\mathcal{V}_\mathrm{s} = \{n_\mathrm{r} + 1, \ldots, n_\mathrm{r} + n_\mathrm{s}\}$
- Link probability matrix for edges between regular agents: $\Psi^{(\mathrm{r})} = (\Psi^{(\mathrm{r})})^\mathsf{T} = [\psi_{ij}^{(\mathrm{r})}] \in [0,1]^{n_\mathrm{r} \times n_\mathrm{r}}$
- Link probability matrix for edges between regular and stubborn agents: $\Psi^{(\mathrm{s})} = [\psi_{ij}^{(\mathrm{s})}] \in [0,1]^{n_\mathrm{r} \times n_\mathrm{s}}$

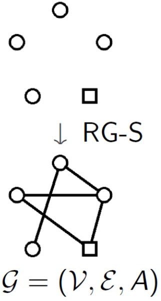
In the **random graph model with stubborn agents** RG-S$(n_\mathrm{r}, n_\mathrm{s}, \Psi^{(\mathrm{r})}, \Psi^{(\mathrm{s})})$, a random graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ is constructed

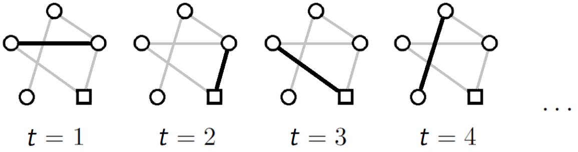
(i) by generating a random graph on the regular agents from RG$(n_\mathrm{r}, \Psi^{(\mathrm{r})})$,

(ii) by adding $\{i, j\}$ to $\mathcal{E}$ with prob. $\psi_{i,j-n_\mathrm{r}}^{(\mathrm{s})}$ independently for $i \in \mathcal{V}_\mathrm{r}$, $j \in \mathcal{V}_\mathrm{s}$.

Gossip Dynamics over RG-S

The gossip dynamics over RG-S is an opinion dynamics evolving over a random graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ generated from an RG-S.





$$X_i(t+1) = \begin{cases} \frac{1}{2}(X_i(t) + X_j(t)), & \text{if } j \text{ is regular,} \\ \frac{1}{2}(X_i(t) + z_j^{(s)}), & \text{if } j \text{ is stubborn.} \end{cases}$$

Remark: Two sources of randomness

1. Random graph $\mathcal{G}$ is constructed from an RG-S.
2. (Random) Gossip dynamics evolve over a realization of the random graph $\mathcal{G}$.

Gossip Dynamics over a Graph $\mathcal{G}$

Suppose $\mathcal{G}$ with n agents is connected.

- If there is no stubborn agent, regular agents reach consensus [Boyd et al., '06].

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- The distance of current average from initial average $|\mathbf{1}^\top X(t)/n - \mathbf{1}^\top X(0)/n|$ concentrates around zero [Fagnani & Zampieri, '08; Vanka et al., '09].

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- The distance of current average from initial average $|\mathbf{1}^\top X(t)/n - \mathbf{1}^\top X(0)/n|$ concentrates around zero [Fagnani & Zampieri, '08; Vanka et al., '09].
- If there are stubborn agents with different opinions, $X(t)$ keeps fluctuating [Acemoğlu et al.,'13], but
  (i) $X(t)$ converges in distribution to a unique stationary distribution with mean $\mathbf{x}^{\mathcal{G},n}$.
  (ii) The time average $S(t) = \frac{1}{t}\sum_{i=0}^{t-1} X(i)$ converges, and $\lim_{t\to\infty} S(t) = \mathbf{x}^{\mathcal{G},n}$ a.s.

Gossip Dynamics over a Graph $\mathcal{G}$

The following results hold for the expected final opinions $\mathbf{x}^{\mathcal{G},n}$.

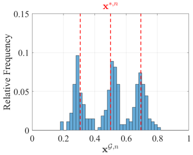
(i) A network is called highly fluid, if the product of the mixing time of random walks on that network and the aggregate centrality of stubborn agents is small.

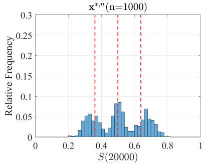
If the network is highly fluid, the entries of $\mathbf{x}^{\mathcal{G},n}$ concentrate around a fixed value [Acemoğlu et al., '13].

(ii) If regular agents form two communities connected to different stubborn agents and the influence of stubborn agents is large, $\mathbf{x}^{\mathcal{G},n}$ polarize accordingly [Como & Fagnani, '16].

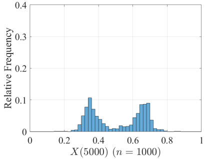
Problem Formulation

1. Given a gossip opinion dynamics over an RG-S, provide high-probability bounds for the difference between the **expected final opinions** $\mathbf{x}^{\mathcal{G},n}$ and the **expected final opinions over an averaged graph** $\mathbf{x}^{*,n}$.





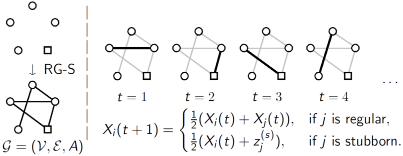
2. Provide high-probability bounds for the difference between the **opinion time average** $S(t)$ and the **expected final opinions** $\mathbf{x}^{*,n}$.



3. Given a gossip opinion dynamics over an RG-S with **community structure**, provide bounds for the difference between opinions $X(t)$ and expected average opinions over finite time intervals.

Gossip Dynamics over Averaged Graph

Gossip dynamics over RG-S has expected final opinions $\mathbf{x}^{\mathcal{G},n} = \lim_{t\to\infty} \mathbb{E}_{\mathcal{G}}\{X(t)\}$.



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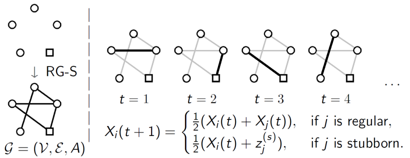
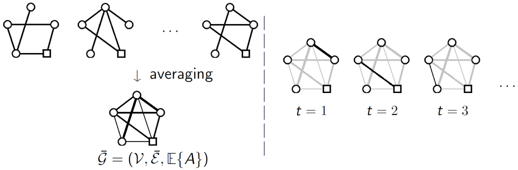


The averaged graph $\bar{\mathcal{G}} = (\mathcal{V}, \bar{\mathcal{G}}, \mathbb{E}\{A\})$ is obtained by averaging random graph $\mathcal{G}$.
Consider gossip dynamics over $\bar{\mathcal{G}}$.



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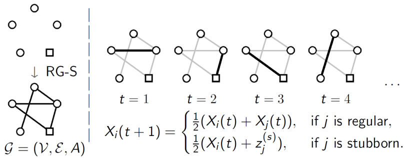

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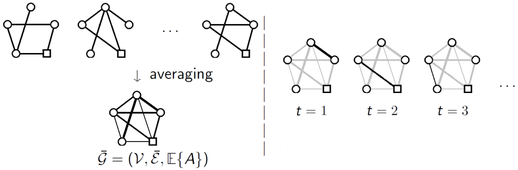

Assumption 1

The stubborn agents of the averaged graph $\bar{\mathcal{G}}$ have the same opinions $z^{(s)}$ as the stubborn agents of RG-S.

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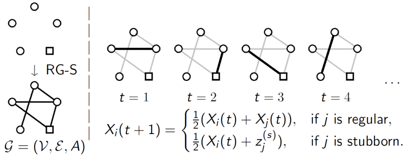

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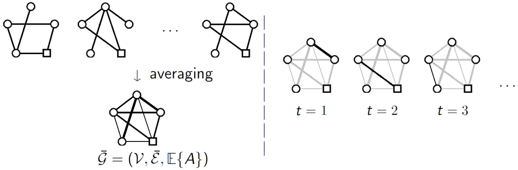

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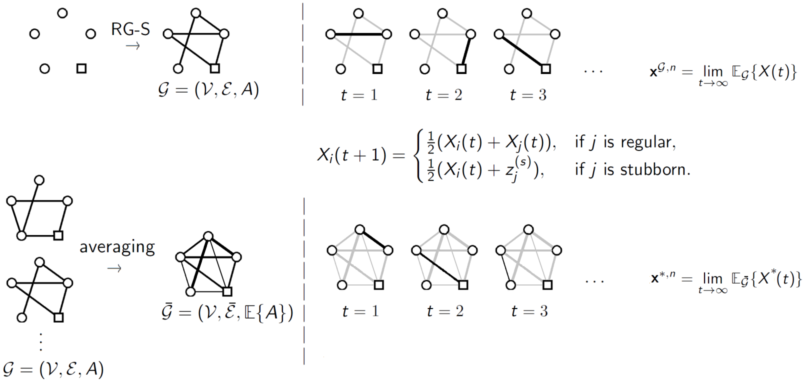
The expected final opinions over the average graph are denoted by $\mathbf{x}^{*,n}$.

Concentration Inequality of Expected Final Opinions

Theorem 1 (Informal)

If the influence of stubborn agents is large enough, then expected final opinions $\mathbf{x}^{\mathcal{G},n}$ are close to the expected final opinions over the averaged graph $\mathbf{x}^{*,n}$ with high probability.



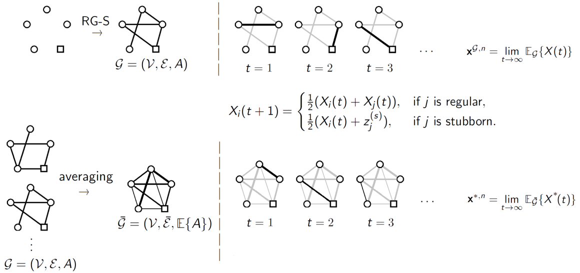
Theorem 1

If $\delta_{\mathsf{rs}} > 8\log n$, then for a positive constant c

$$\mathbb{P}\{\|\mathbf{x}^{\mathcal{G},n} - \mathbf{x}^{*,n}\| \le \varepsilon_{x,n}\|z^{(\mathsf{s})}\|\} \ge 1 - n^{-c},$$

$$\varepsilon_{x,n} = 4\left(\frac{\sqrt{\max\{\Delta_{\mathsf{rs}}, \Delta_{\mathsf{sr}}\}\log n}}{\delta_{\mathsf{rs}}} + \frac{2\max\{\Delta_{\mathsf{rs}}, \Delta_{\mathsf{sr}}\}\sqrt{\Delta_{\mathsf{r}}\log n}}{\delta_{\mathsf{rs}}^2}\right).$$

$\delta_{\mathsf{rs}} = \min_{i\in\mathcal{V}_{\mathsf{r}}}\{\mathbb{E}\{\sum_{j\in\mathcal{V}_{\mathsf{s}}} a_{ij}\}\}$, $\Delta_{\mathsf{rs}} = \max_{i\in\mathcal{V}_{\mathsf{r}}}\{\mathbb{E}\{\sum_{j\in\mathcal{V}_{\mathsf{s}}} a_{ij}\}\}$, $\Delta_{\mathsf{sr}} = \max_{i\in\mathcal{V}_{\mathsf{s}}}\{\mathbb{E}\{\sum_{j\in\mathcal{V}_{\mathsf{r}}} a_{ij}\}\}$, $\Delta_{\mathsf{r}} = \max_{i\in\mathcal{V}_{\mathsf{r}}}\{\mathbb{E}\{\sum_{j\in\mathcal{V}} a_{ij}\}\}$.



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Remark

- The result provides a bound for $\|\mathbf{x}^{\mathcal{G},n} - \mathbf{x}^{*,n}\|$ with high probability w.r.t. the randomness of $\mathcal{G}$.

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- The result provides a bound for $\|\mathbf{x}^{\mathcal{G},n} - \mathbf{x}^{*,n}\|$ with high probability w.r.t. the randomness of $\mathcal{G}$.
- The bound is related to expected number of edges between agents.
- The sharpness of concentration is influenced by the minimum expected influence of stubborn agents on a regular agent.
- If $\delta_{\mathrm{rs}} = 0$ (some agents not directly influenced by stubborn agents), then other expressions of $\varepsilon_{x,n}$ can be derived.

Proof Sketch

1. Write the gossip dynamics over $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ in compact form,

$$X(t+1) = Q(t)X(t) + R(t)z^{(\mathrm{s})}, \quad \{Q(t), R(t)\} \text{ i.i.d. random matrices.}$$

   So $\mathbf{x}^{\mathcal{G},n} = \lim_{t\to\infty} \mathbb{E}_{\mathcal{G}}\{X(t)\} = (I_{n_\mathrm{r}} - \mathbb{E}_{\mathcal{G}}\{Q(t)\})^{-1}\mathbb{E}_{\mathcal{G}}\{R(t)\}z^{(\mathrm{s})}$, where

$$\mathbb{E}_{\mathcal{G}}\{Q(t)\} = I_{n_\mathrm{r}} - \frac{1}{2\alpha}\begin{bmatrix} \sum_{i=1}^{n} a_{1i} & -a_{12} & \cdots & -a_{1,n_\mathrm{r}} \\ -a_{21} & \sum_{i=1}^{n} a_{2i} & & \vdots \\ \vdots & & \ddots & -a_{n_\mathrm{r}-1,n_\mathrm{r}} \\ -a_{n_\mathrm{r},1} & \cdots & -a_{n_\mathrm{r},n_\mathrm{r}-1} & \sum_{i=1}^{n} a_{n_\mathrm{r},i} \end{bmatrix} =: I_{n_\mathrm{r}} - \frac{\bar{M}}{2\alpha},$$

$$\mathbb{E}_{\mathcal{G}}\{R(t)\} = \frac{1}{2\alpha}\begin{bmatrix} a_{1,n_\mathrm{r}+1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n_\mathrm{r},n_\mathrm{r}+1} & \cdots & a_{n_\mathrm{r},n} \end{bmatrix} =: \frac{\bar{U}}{2\alpha}, \quad \alpha = |\mathcal{E}|.$$

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2. For gossip dynamics over averaged graph $X^*(t+1) = Q^*(t)X^*(t) + R^*(t)z^{(\mathrm{s})}$, similarly we have $\mathbf{x}^{*,n} = \lim_{t\to\infty} \mathbb{E}\{X^*(t)\} = (I_{n_\mathrm{r}} - \mathbb{E}\{Q^*(t)\})^{-1}\mathbb{E}\{R^*(t)\}z^{(\mathrm{s})}$, and

$$\mathbb{E}\{Q^*(t)\} = I_{n_\mathrm{r}} - \frac{\mathbb{E}\{\bar{M}\}}{2\mathbb{E}\{\alpha\}}, \quad \mathbb{E}\{R^*(t)\} = \frac{\mathbb{E}\{\bar{U}\}}{2\mathbb{E}\{\alpha\}} = \frac{\Psi^{(\mathrm{s})}}{2\mathbb{E}\{\alpha\}}.$$

Proof Sketch

1. $\mathbf{x}^{\mathcal{G},n} = \left(\frac{\bar{M}}{2\alpha}\right)^{-1} \frac{\bar{U}}{2\alpha} z^{(\mathsf{s})}.$

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3. Apply matrix concentration inequalities to random matrices $\bar{M}$ and $\bar{U}$ and Chernoff's bound to α, to bound $\|\bar{M} - \mathbb{E}\{\bar{M}\}\|$, $\|\bar{U} - \Psi^{(\mathrm{s})}\|$, and $|\alpha - \mathbb{E}\{\alpha\}|$.
   Then apply the matrix perturbation inequality $\|C^{-1} - D^{-1}\| \le \|C^{-1}\| \, \|D^{-1}\| \|C - D\|$ to get an upper bound for $\|\bar{M}^{-1} - \mathbb{E}\{\bar{M}\}^{-1}\|$.

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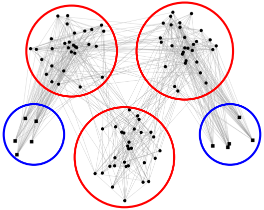
4. The conclusion follows from

$$
\begin{aligned}
&\|\mathbf{x}^{\mathcal{G},n} - \mathbf{x}^{*,n}\| \\
&\leq \|\bar{M}^{-1}\bar{U} - \mathbb{E}\{\bar{M}\}^{-1}\Psi^{(s)}\| \|z^{(s)}\| \\
&\leq (\|\bar{M}^{-1}\| \|\bar{U} - \Psi^{(s)}\| + \|\bar{M}^{-1} - \mathbb{E}\{\bar{M}\}^{-1}\| \|\Psi^{(s)}\|) \|z^{(s)}\|.
\end{aligned}
$$

Example: Concentration of Expected Final Opinions

RG-S has three communities with regular agents and two communities with stubborn agents.



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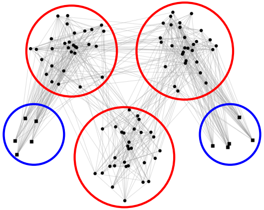
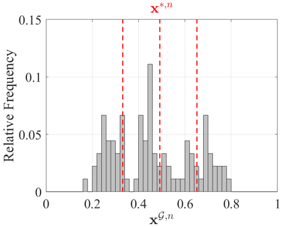
$\mathbf{x}^{*,n}$ has three distinct values.



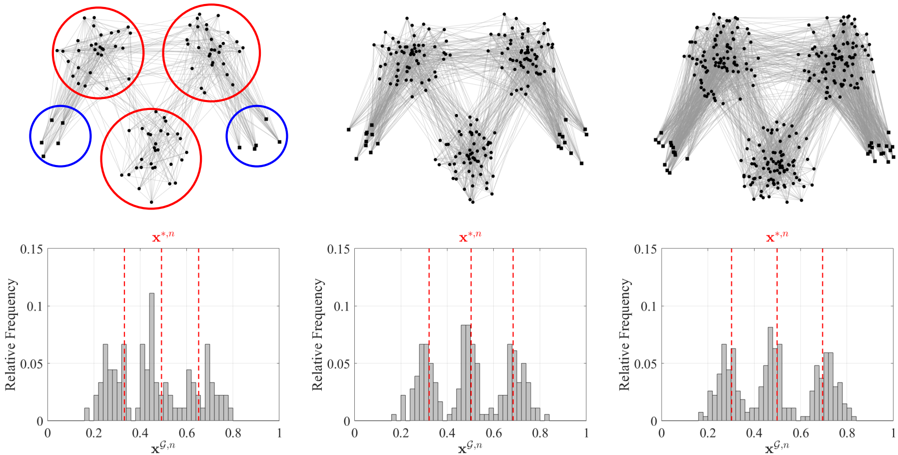


$n = 100$

Example: Concentration of Expected Final Opinions

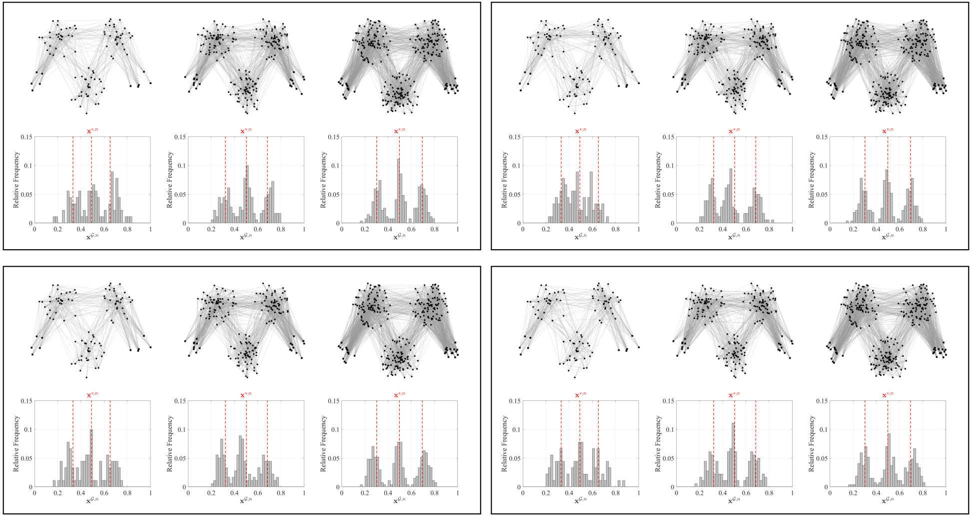
RG-S has three communities with regular agents and two communities with stubborn agents.

$\mathbf{x}^{*,n}$ has three distinct values. Concentration appears as network size n increases.



$n = 100$ $n = 200$ $n = 300$

When n is large, edges of sampled graphs can be different but expected final opinions are close to the averaged version.



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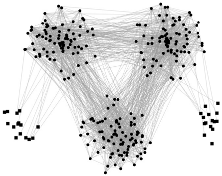
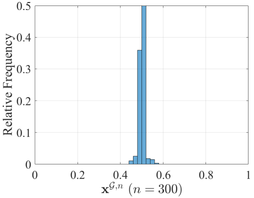
The behavior of $\mathbf{x}^{\mathcal{G},n}$ can be understood by studying $\mathbf{x}^{*,n}$, which is easier to analyze.

Theorem 2 (Informal)

(i) (When stubborn agents have relatively small influence) If the influence of stubborn agents is large enough for concentration to hold, but the connectivity between regular agents is much larger, then expected final opinion $\mathbf{x}^{\mathcal{G},n}$ is close to a consensus vector with high probability.





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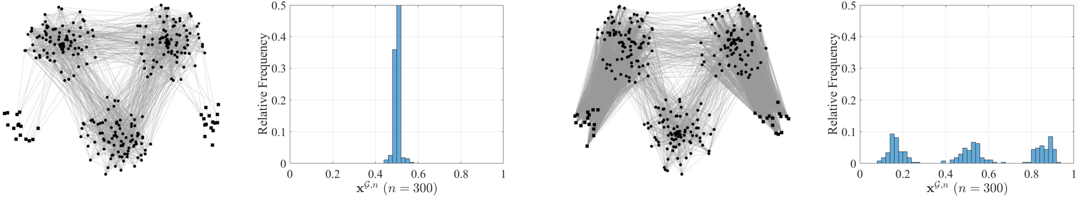
(i) (When stubborn agents have relatively small influence) If the influence of stubborn agents is large enough for concentration to hold, but the connectivity between regular agents is much larger, then expected final opinion $\mathbf{x}^{\mathcal{G},n}$ is close to a consensus vector with high probability.

(ii) (When stubborn agents have relatively large influence) If the influence of stubborn agents is large enough for concentration to hold, and the influence of stubborn agents is much larger than connectivity between regular agents, then each agent's expected final opinion is close to a weighted average of stubborn agent opinions.



Theorem 2

(i) (When stubborn agents have relatively small influence)
If $\min\{\Delta_{\mathrm{rs}}, \Delta_{\mathrm{sr}}\} \geq \log n$, $\lambda_1(\mathbb{E}\{\bar{M}\}) = \omega(\max\{\sqrt{\Delta_{\mathrm{r}} \log n}, \Delta_{\mathrm{rs}}, \Delta_{\mathrm{sr}}\})$, and $\lambda_2(\mathbb{E}\{\bar{L}\}) = \omega(\max\{\Delta_{\mathrm{rs}}, \Delta_{sr}\})$, where $\bar{L}$ is the Laplacian matrix of the subgraph induced by regular agents, then there exists γ s.t.

$$\mathbb{P}\{\|\mathbf{x}^{\mathcal{G},n} - \gamma \mathbf{1}_{n_{\mathrm{r}}}\| \leq o(\|z^{(\mathrm{s})}\|)\} = 1 - o(1).$$

$f(n) = \omega(g(n))$, if $f(n)/g(n) \to \infty$ as $n \to \infty$.
$\delta_{\mathrm{rs}} = \min_{i \in \mathcal{V}_{\mathrm{r}}} \{\mathbb{E}\{\sum_{j \in \mathcal{V}_{\mathrm{s}}} a_{ij}\}\}$, $\Delta_{\mathrm{rs}} = \max_{i \in \mathcal{V}_{\mathrm{r}}} \{\mathbb{E}\{\sum_{j \in \mathcal{V}_{\mathrm{s}}} a_{ij}\}\}$, $\Delta_{\mathrm{sr}} = \max_{i \in \mathcal{V}_{\mathrm{s}}} \{\mathbb{E}\{\sum_{j \in \mathcal{V}_{\mathrm{r}}} a_{ij}\}\}$, $\Delta_{\mathrm{r}} = \max_{i \in \mathcal{V}_{\mathrm{r}}} \{\mathbb{E}\{\sum_{j \in \mathcal{V}} a_{ij}\}\}$.

Theorem 2

(i) (When stubborn agents have relatively small influence)
If $\min\{\Delta_{\text{rs}}, \Delta_{\text{sr}}\} \geq \log n$, $\lambda_1(\mathbb{E}\{\bar{M}\}) = \omega(\max\{\sqrt{\Delta_{\text{r}} \log n}, \Delta_{\text{rs}}, \Delta_{\text{sr}}\})$, and $\lambda_2(\mathbb{E}\{\bar{L}\}) = \omega(\max\{\Delta_{\text{rs}}, \Delta_{sr}\})$, where $\bar{L}$ is the Laplacian matrix of the subgraph induced by regular agents, then there exists γ s.t.

$$\mathbb{P}\{\|\mathbf{x}^{\mathcal{G},n} - \gamma \mathbf{1}_{n_{\text{r}}}\| \leq o(\|z^{(\text{s})}\|)\} = 1 - o(1).$$

(ii) (When stubborn agents have relatively large influence)
If $\delta_{\text{rs}} = \omega(\sqrt{(\Delta_{\text{r}} \log n)^{1/2} \max\{\Delta_{\text{rs}}, \Delta_{\text{sr}}\}})$ and $\Delta_{\text{rr}} = o(\delta_{\text{rs}}^2 / \max\{\Delta_{\text{rs}}, \Delta_{\text{sr}}\})$, then

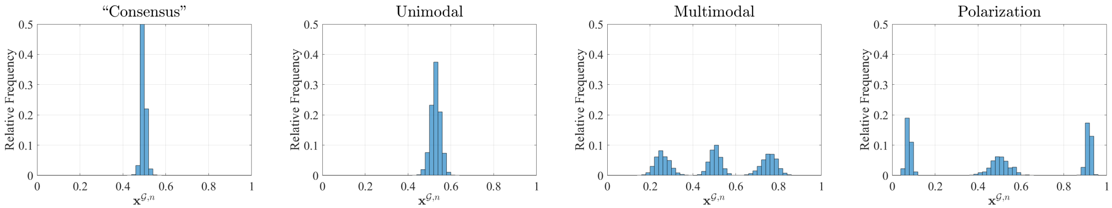
$$\mathbb{P}\{\|\mathbf{x}^{\mathcal{G},n} - (\text{diag}(\Psi^{(\text{s})}\mathbf{1}_{n_{\text{s}}}))^{-1}\Psi^{(\text{s})}z^{(\text{s})}\| \leq o(\|z^{(\text{s})}\|)\} = 1 - o(1).$$

Here $\Psi^{(\text{s})} = [\psi_{ij}^{(\text{s})}] \in [0,1]^{n_{\text{r}} \times n_{\text{s}}}$ is the link probability matrix for edges between regular and stubborn agents.

$f(n) = \omega(g(n))$, if $f(n)/g(n) \to \infty$ as $n \to \infty$.
$\delta_{\text{rs}} = \min_{i \in \mathcal{V}_{\text{r}}} \{\mathbb{E}\{\sum_{j \in \mathcal{V}_{\text{s}}} a_{ij}\}\}$, $\Delta_{\text{rs}} = \max_{i \in \mathcal{V}_{\text{r}}} \{\mathbb{E}\{\sum_{j \in \mathcal{V}_{\text{s}}} a_{ij}\}\}$, $\Delta_{\text{sr}} = \max_{i \in \mathcal{V}_{\text{s}}} \{\mathbb{E}\{\sum_{j \in \mathcal{V}_{\text{r}}} a_{ij}\}\}$, $\Delta_{\text{r}} = \max_{i \in \mathcal{V}_{\text{r}}} \{\mathbb{E}\{\sum_{j \in \mathcal{V}} a_{ij}\}\}$.

From “Consensus” to Polarization

The network structure (the influence of stubborn agents and connectivity between regular agents) affects expected final opinions.



Summary

The derived concentration inequality is able to predict various types of opinion distributions.

Stubborn agents have

- small influence: expected final opinions are close to each other
- moderate influence: opinion distributions have multiple modes corresponding to a community structure
- large influence: regular agents highly influenced by stubborn agents polarize; other agents have opinions depending on network structure

Concentration Inequality of Opinion Time Average

Opinions $X(t)$, instead of their expectations, can be observed in practice.
Recall that the opinion time average $S(t) = \frac{1}{t}\sum_{i=0}^{t-1} X(i)$ converges to the expected final opinions $\mathbf{x}^{\mathcal{G},n}$ over time.

Theorem 3 (Informal)

Suppose that the influence of stubborn agents is large enough for concentration to hold. Then for large enough t depending on network size n, the opinion time average $S(t)$ is close to $\mathbf{x}^{*,n}$ with high probability.

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Theorem 3

If $\delta_{\mathrm{rs}} = \omega(\sqrt{(\Delta_{\mathrm{r}} \log n)^{1/2}(\Delta_{\mathrm{rs}} \vee \Delta_{\mathrm{sr}})})$, then for $t > t_n$

$$\mathbb{P}\{\|S(t) - \mathbf{x}^{*,n}\| \leq o(\sqrt{n})\} \geq 1 - c_1 n \exp\{-t/t_n^{2+\eta}\} - n^{-c_2},$$

where $t_n = c_3\sqrt{n}\mathbb{E}\{|\mathcal{E}|\}/\delta_{\mathrm{rs}}$, $\eta > 0$, and c_1, c_2, c_3 are universal constants.

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- For fixed n, $c_1 n \exp\{-t/t_n^{2+\eta}\} \to 0$ as $t \to \infty$. $S(t)$ converges to $\mathbf{x}^{\mathcal{G},n}$ concentrating around $\mathbf{x}^{*,n}$ with failure probability n^{-c_2}.

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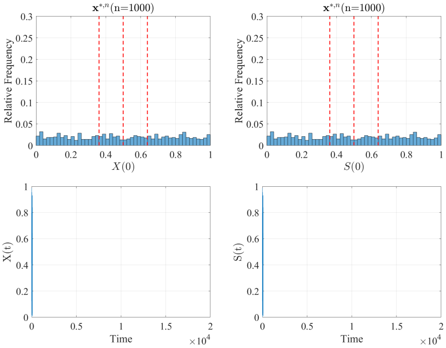
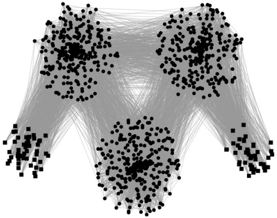
- For large n, the failure probability mainly depends on time t, which must be large enough to ensure a nontrivial bound.
- Similar results hold for the case where $\delta_{\mathrm{rs}} = 0$.
- Proof technique: Theorem 1 provides bounds for $\|\mathbf{x}^{\mathcal{G},n} - \mathbf{x}^{*,n}\|$, so it suffices to bound $\|S(t) - \mathbf{x}^{\mathcal{G},n}\|$. A Hoeffding's inequality for Markov chains is derived using martingale methods and applied to the analysis.

Example: Concentration of Opinion Time Average

Opinion time average $S(t)$ quickly concentrates around $\mathbf{x}^{*,n}$, but agent opinion vector $X(t)$ does not have this clear pattern.



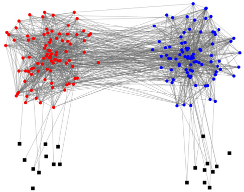


In the second row, a small number of agents are selected for the illustration of the dynamics.

Concentration Inequality of Transient Opinions

It could take a long time for large-scale networked dynamics to reach steady state, so it is relevant to study transient behavior.

Assumption 2 (Community structure)



(i) RG-S$(n_\mathrm{r}, n_\mathrm{s}, \Psi^{(\mathrm{r})}, \Psi^{(\mathrm{s})})$ satisfies that n_r is even, and

$$\Psi^{(\mathrm{r})} = \begin{bmatrix} \mathbf{1}_{n_\mathrm{r}/2} & \mathbf{0}_{n_\mathrm{r}/2} \\ \mathbf{0}_{n_\mathrm{r}/2} & \mathbf{1}_{n_\mathrm{r}/2} \end{bmatrix} \begin{bmatrix} \psi_\mathrm{s}^{(\mathrm{r})} & \psi_\mathrm{d}^{(\mathrm{r})} \\ \psi_\mathrm{d}^{(\mathrm{r})} & \psi_\mathrm{s}^{(\mathrm{r})} \end{bmatrix} \begin{bmatrix} \mathbf{1}_{n_\mathrm{r}/2}^\mathsf{T} & \mathbf{0}_{n_\mathrm{r}/2}^\mathsf{T} \\ \mathbf{0}_{n_\mathrm{r}/2}^\mathsf{T} & \mathbf{1}_{n_\mathrm{r}/2}^\mathsf{T} \end{bmatrix}$$

where $\psi_\mathrm{s}^{(\mathrm{r})}$ is the link probability for agents in the same community,
and $\psi_\mathrm{d}^{(\mathrm{r})}$ is the link probability for agents in different communities.
(ii) Assume for simplicity that there exists $\psi^{(\mathrm{s})} \geq 0$ s.t. $\Psi^{(\mathrm{s})}\mathbf{1}_{n_\mathrm{s}} = \psi^{(\mathrm{s})}\mathbf{1}_{n_\mathrm{s}}$. Denote the average link probability by $\psi_0^{(\mathrm{s})} := \psi^{(\mathrm{s})}/n_\mathrm{r}$.

Assumption 3 (Bounded initial condition)

There exists $c_x > 0$ s.t. $|X_i(0)| < c_x$ and $|z_j^{(\mathrm{s})}| < c_x$, $\forall i \in \mathcal{V}_\mathrm{r}$, $j \in \mathcal{V}_\mathrm{s}$. That is, the system is bounded.

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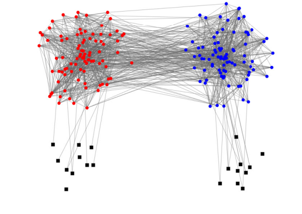

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Remark

Note RG$(n_\mathrm{r}, \Psi^{(\mathrm{r})})$ is a stochastic block model with two equal-sized communities [Abbe, '17].

Concentration Inequality of Transient Opinions

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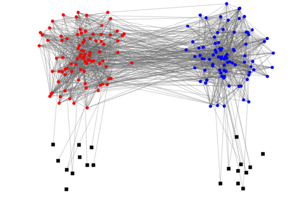


(ii) Assume for simplicity that there exists $\psi^{(\mathrm{s})} \geq 0$ s.t. $\Psi^{(\mathrm{s})}\mathbf{1}_{n_\mathrm{s}} = \psi^{(\mathrm{s})}\mathbf{1}_{n_\mathrm{s}}$. $\psi_0^{(\mathrm{s})} := \psi^{(\mathrm{s})}/n_\mathrm{r}$.

(i) Two communities with regular agents $\mathcal{V}_{\mathrm{r}1} = \{1, \ldots, n_\mathrm{r}/2\}$ and $\mathcal{V}_{\mathrm{r}2} = \{1 + n_\mathrm{r}/2, \ldots, n_\mathrm{r}\}$.

Expected average opinions within communities $\chi_k(t) := \frac{2}{n_\mathrm{r}} \sum_{j \in \mathcal{V}_{\mathrm{r}k}} \mathbb{E}_\mathcal{G}\{X_j(t)\}$,

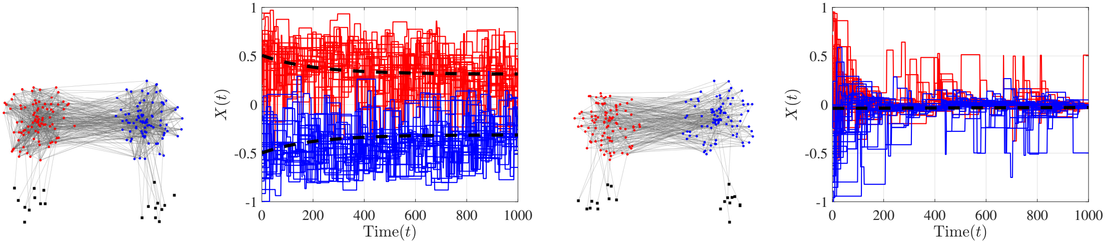
(ii) Expected average opinion over the network $\iota(t) := \frac{1}{n_\mathrm{r}} \sum_{j \in \mathcal{V}_\mathrm{r}} \mathbb{E}_\mathcal{G}\{X_j(t)\}$.

Theorem 4 (Informal)

Suppose that the influence of stubborn agents is small.

(i) If the link probability within communities $\psi_{\mathsf{s}}^{(\mathsf{r})}$ is **larger than** between communities $\psi_{\mathsf{d}}^{(\mathsf{r})}$, then most agents have opinions close to expected average opinions **within their communities** over a transient interval.

(ii) If the link probability within communities $\psi_{\mathsf{s}}^{(\mathsf{r})}$ is **similar to or smaller than** between communities $\psi_{\mathsf{d}}^{(\mathsf{r})}$, then most agents have opinions close to expected average opinion **over the network**.



Theorem 4

(i) If $\psi_{\mathrm{s}}^{(\mathrm{r})} = \omega(\max\{\psi_{\mathrm{d}}^{(\mathrm{r})} \log n, (\log n)^3/n\})$ and $\psi_{\mathrm{d}}^{(\mathrm{r})} = \omega(\psi_0^{(\mathrm{s})} \log n)$, then for $\varepsilon > 0$ and $t \in (\Theta(n \log n), o(\min\{n\psi_{\mathrm{s}}^{(\mathrm{r})}/\psi_{\mathrm{d}}^{(\mathrm{r})}, n\sqrt{\psi_{\mathrm{s}}^{(\mathrm{r})} n/(\log n)}\}))$,

$$\mathbb{P}\{\#\{i : |X_i(t) - \chi_{\mathcal{C}_i}(t)| > \varepsilon c_x\} = o(n)\} = 1 - o(1),$$

where $\mathcal{C}_i = k$ if $i \in \mathcal{V}_{\mathrm{r}k}$, $k = 1, 2$, and $\chi_k(t) = (2/n_{\mathrm{r}}) \sum_{j \in \mathcal{V}_{\mathrm{r}k}} \mathbb{E}_{\mathcal{G}}\{X_j(t)\}$.

$f(n) = \omega(g(n))$, if $f(n)/g(n) \to \infty$ as $n \to \infty$.
$f(n) = \Theta(g(n))$, if $f(n), g(n) > 0$ and $\exists c_1, c_2$ s.t. $c_1 g(n) < f(n) < c_2 g(n)$.
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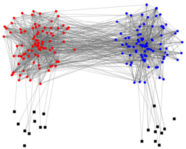

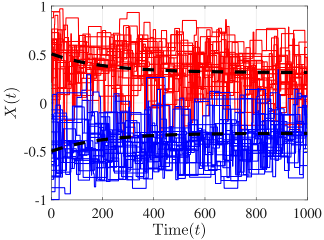

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$$\mathbb{P}\{\#\{i : |X_i(t) - \chi_{\mathcal{C}_i}(t)| > \varepsilon c_x\} = o(n)\} = 1 - o(1),$$

where $\mathcal{C}_i = k$ if $i \in \mathcal{V}_{\mathrm{r}k}$, $k = 1, 2$, and $\chi_k(t) = (2/n_{\mathrm{r}}) \sum_{j \in \mathcal{V}_{\mathrm{r}k}} \mathbb{E}_{\mathcal{G}}\{X_j(t)\}$.

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Remark

When stubborn agents have small influence ($\psi_0^{(\mathrm{s})} = o((\log n)/n)$ is allowed) and edges within communities are denser than between communities, agent opinions concentrate around averages within communities.

The length of the time interval depends on relative edge density.

Theorem 4

(i) If $\psi_s^{(r)} = \omega(\max\{\psi_d^{(r)} \log n, (\log n)^3/n\})$ and $\psi_d^{(r)} = \omega(\psi_0^{(s)} \log n)$, then for $\varepsilon > 0$ and $t \in (\Theta(n \log n), o(\min\{n\psi_s^{(r)}/\psi_d^{(r)}, n\sqrt{\psi_s^{(r)} n/(\log n)}\}))$,

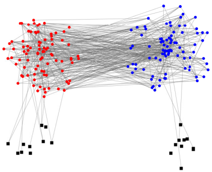
$$\mathbb{P}\{\#\{i : |X_i(t) - \chi_{\mathcal{C}_i}(t)| > \varepsilon c_x\} = o(n)\} = 1 - o(1),$$

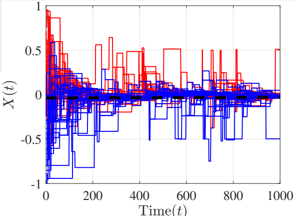
where $\mathcal{C}_i = k$ if $i \in \mathcal{V}_{rk}$, $k = 1, 2$, and $\chi_k(t) = (2/n_r)\sum_{j \in \mathcal{V}_{rk}} \mathbb{E}_{\mathcal{G}}\{X_j(t)\}$.

(ii) If $\psi_s^{(r)} < \psi_d^{(r)} + o(\psi_d^{(r)})$, $\psi_d^{(r)} = \omega(\psi_0^{(s)} \log n)$, and $\psi_s^{(r)} = \omega((\log n)^3/n)$, then for $\varepsilon > 0$ and $t \in \left(\Theta(n \log n), o(\min\{n\psi_d^{(r)}/\psi_0^{(s)}, n\sqrt{\psi_d^{(r)} n/(\log n)}\})\right)$,

$$\mathbb{P}\{\#\{i : |X_i(t) - \iota(t)| > \varepsilon c_x\} = o(n)\} = 1 - o(1),$$

where $\iota(t) = (1/n_r)\sum_{j \in \mathcal{V}_r} \mathbb{E}_{\mathcal{G}}\{X_j(t)\}$.





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where $\mathcal{C}_i = k$ if $i \in \mathcal{V}_{\mathrm{r}k}$, $k = 1, 2$, and $\chi_k(t) = (2/n_{\mathrm{r}}) \sum_{j \in \mathcal{V}_{\mathrm{r}k}} \mathbb{E}_{\mathcal{G}}\{X_j(t)\}$.

(ii) If $\psi_{\mathrm{s}}^{(\mathrm{r})} < \psi_{\mathrm{d}}^{(\mathrm{r})} + o(\psi_{\mathrm{d}}^{(\mathrm{r})})$, $\psi_{\mathrm{d}}^{(\mathrm{r})} = \omega(\psi_0^{(\mathrm{s})} \log n)$, and $\psi_{\mathrm{s}}^{(\mathrm{r})} = \omega((\log n)^3/n)$, then for $\varepsilon > 0$ and $t \in \big(\Theta(n \log n), o(\min\{n\psi_{\mathrm{d}}^{(\mathrm{r})}/\psi_0^{(\mathrm{s})}, n\sqrt{\psi_{\mathrm{d}}^{(\mathrm{r})} n/(\log n)}\}))$,

$$\mathbb{P}\{\#\{i : |X_i(t) - \iota(t)| > \varepsilon c_x\} = o(n)\} = 1 - o(1),$$

where $\iota(t) = (1/n_{\mathrm{r}}) \sum_{j \in \mathcal{V}_{\mathrm{r}}} \mathbb{E}_{\mathcal{G}}\{X_j(t)\}$.

Remark

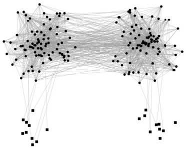
When stubborn agents have small influence, and edges between communities are denser than or similar to edges within communities, opinions concentrate around the opinion average over the graph.

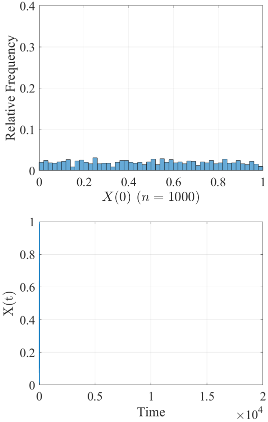
Example: Transient Behavior

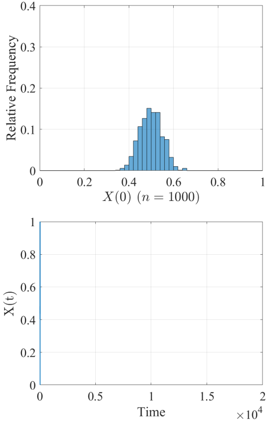
Similar opinion profile (unimodal, bimodal,...) can appear at different phases of a process.

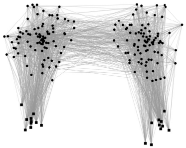
When stubborn agents have small influence, regular agents first form two clusters but eventually one cluster (recall expected final opinions in Thm 2).

When stubborn agents have large influence, regular agents starting with one cluster split into two clusters that align with influential stubborn agents.









Application: Community Detection

Suppose we only observe a trajectory $X(t)$ of the gossip dynamics, and we do not know the underlying network $\mathcal{G}$. Can we recover the community structure?

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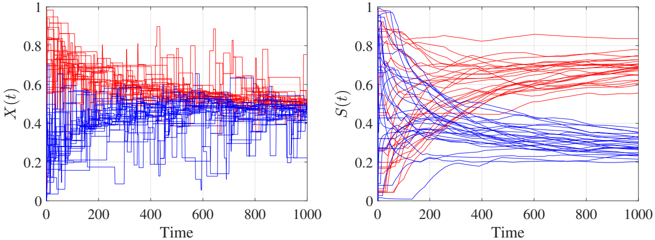


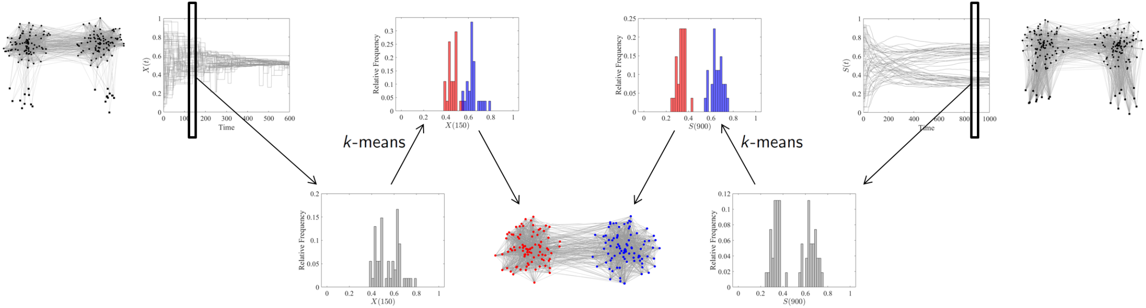
Simple Detection Algorithms

Detection algorithm based on transient opinions

1. Select transient opinions $X(t)$ at time t.
2. Apply k-means with $k = 2$ to $X(t)$ to estimate the community structure by paritioning $\{1, \ldots, n_r\}$ into two subgroups.

Detection algorithm based on time-averaged opinions

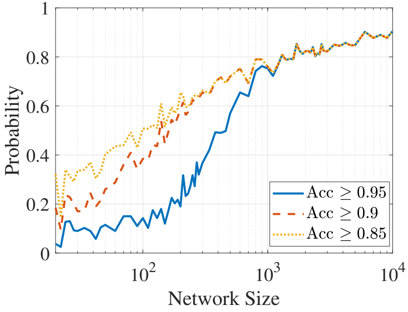
1. Compute time-averaged opinions $S(t) = \sum_{i=1}^{t} X(i)$.
2. Apply k-means with $k = 2$ to $S(t)$ to estimate the community structure by paritioning $\{1, \ldots, n_r\}$ into two subgroups.



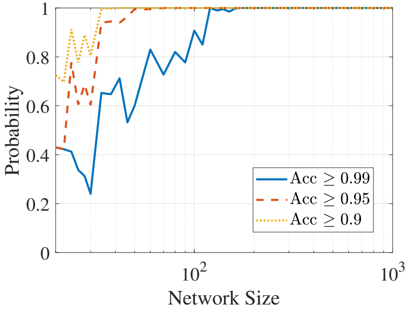
Simulation: Community Detection from Opinion Dynamics

Detection algorithms based on transient or time-averaged opinions can recover most community labels. The accuracy of an algorithm is (# correctly classfied agents)/(# agents). The probability of Acc larger than a given value increases as n grows.

When the influence of stubborn agents is small, detection based on clustering transient opinions $X([n \log n])$ achieves almost exact recovery.



When the influence of stubborn agents is large, detection based on clustering time-averaged opinions $S([n(\log n)^{2.5}])$ achieves almost exact recovery.



$[\cdot]$ is the rounding function.

Conclusions

- Analyzed gossip dynamics over random graphs with stubborn agents.
- Concentration inequalities for expected final opinions, time-averaged opinions, and transient opinions.
- Explicit dependencies on time and network size
- Application to a simple community detection problem.

Future work: Derive concentration inequalities for other networked dynamics, e.g., FJ model, continuous-opinion-discrete-action models, nonlinear models.

`https://people.kth.se/~kallej/`