Dynamics and Balance on Signed Networks

R. Lambiotte

renaud.lambiotte@maths.ox.ac.uk
A network science approach to signed graphs
Change of perspective

Compared with mean-field approaches, which summarise interactions between all elements with a single averaged field, network models often have much higher explanatory power because they can account for sparse and non-random topologies.

To capture direct interactions, a network represents components with nodes and pairwise interactions. To capture indirect interactions, one defines the notion of walk/path/connectivity.
Data, Dynamics and structure

**DATA**

Fuel for the theoretical modelling

Effect of topology on spreading:
What are the network properties that slow down or accelerate the dynamics?

Uncover structure from dynamics:
What are the important nodes or the important substructures in the graph? Community detection, graph embeddings, etc.
Data collection and community detection
DEBAGREEMENT: A comment-reply dataset for (dis)agreement detection in online debates

John Pougue-Biyong*
University of Oxford
john.pougue-biyong@maths.ox.ac.uk

Valentina Semenova*
University of Oxford
valentina.semenova@maths.ox.ac.uk

Alexandre Matton
Scale AI
alexandre.matton@scale.com

Rachel Han
Scale AI
rachel.han@scale.com

Aerin Kim
Scale AI
aerin.kim@scale.com

Renaud Lambiotte
University of Oxford
renaud.lambiotte@maths.ox.ac.uk

J. Doyne Farmer
University of Oxford
doyne.farmer@inet.ox.ac.uk
- **r/BlackLivesMatter** discusses news related to the *Black Lives Matter* movement. It was created in 2014 and has 109K members,

- **r/Brexit** aims to foster debate about the United Kingdom’s (UK) exit from the European Union (EU). It was created in 2014 and has 53K members,

- **r/climate** is a community for truthful science-based news about climate and related politics and activism. It was created in 2008 and has 99K members,

- **r/democrats** is a partisan subreddit. It aims to discuss political news, policies and how to ensure the election of Democratic party candidates. It was created in 2014 and has 292K members,

- **r/Republican** is a partisan subreddit for Republicans to discuss issues with each other. It was created in 2008 and has 172K members.
**Graph creation**  For each subreddit \( r/\ast \), the resulting set of interactions forms a multi-edge, temporal graph \( G_{r/\ast} \), where nodes are users, and edges represent a comment-reply interaction between two users. One of the unique advantages of DEBAGREEMENT over other datasets is the additional graph interaction information provided about every subreddit.

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**Figure 2: User interface for annotators**
Table 1: Dataset statistics

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<th>r/BLM</th>
<th>r/Republican</th>
<th>r/democrats</th>
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<td>32%</td>
<td>45%</td>
<td>34%</td>
<td>42%</td>
</tr>
<tr>
<td>neutral</td>
<td>29%</td>
<td>28%</td>
<td>22%</td>
<td>25%</td>
<td>22%</td>
</tr>
<tr>
<td>negative</td>
<td>42%</td>
<td>40%</td>
<td>33%</td>
<td>41%</td>
<td>36%</td>
</tr>
</tbody>
</table>

Figure 3: Number of interactions per subreddit (3-month rolling averages)
most active users of each community, we conclude that community A (in brown) is pro-Brexit and communities B, C and D (in black) express sentiments in favor of the UK remaining in the EU. We look further at the main topics of discussion in each community and conclude that: users in community B are interested in the consequences of Brexit on international trade, users in community C discuss the accountability of UK political figures in what they consider ‘a disaster’ for the UK, and users in community D are mostly interested in UK-EU negotiations and the votes in UK parliament.

Figure 4: Polarisation in r/Brexit over time
Optimising signed modularity

To cluster signed networks, the purpose is to place negative edges between communities and positive edges inside communities.

An efficient way based on NG modularity is to consider the signed network as the combination of two networks, defined by the positive and negative edges respectively, and to seek to optimise the difference of their modularities.

\[ Q = Q^+ - Q^- \]

\[ Q = \sum_{i,j} \left[ A_{i,j} - \left( \frac{d_i^+ d_j^+}{2m^+} - \frac{d_i^- d_j^-}{2m^-} \right) \right] \delta(\sigma_i, \sigma_j) \]

Louvain for signed networks

Initial partition
\( C=12, Q=-0.08 \)

After the first pass
\( C=4, Q=0.38 \)

After the second pass
\( C=2, Q=0.45 \)

In the VM step, one tries to move a node to the community of its neighbours:
sensible choice for positive relations, as nodes should be in the community of one of their neighbours, but not for negative relations, as nodes should actually be placed in a community different from that of their negative connections.
Louvain for signed networks

In the VM step, one could try to move tries to move a node to the any community, not only of the neighbours. Works very well but slows down the method drastically (no locality in the optimisation).

Alternative is to try to move a node to the community of its first neighbour for positive edges, and second neighbour for negative edges (“the enemy of my enemy could be my friend”).

https://pypi.python.org/pypi/louvain/
Laplacian for signed networks
Important structures in a signed graph?

Introduced in 1940s and primarily motivated by social and economic networks, a fundamental notion in the study of signed networks is the so-called **structural balance**.

A signed graph is structurally balanced if and only if there is no cycle with an odd number of negative edges, which defines the cycle to be “negative”.

The following theorem provides an alternative interpretation of structural balance in terms of a bipartition of signed graphs.

**Theorem 2.1** (Structure Theorem for Balance [22]). *A signed graph $G$ is structurally balanced if and only if there is a bipartition of the node set into $V = V_1 \cup V_2$ with $V_1$ and $V_2$ being mutually disjoint and one of them being nonempty, s.t. any edge between the two node subsets is negative while any edge within each node subset is positive.*


Laplacian in unsigned and signed networks

An unsigned graph can be encoded by its signed adjacency matrix $A$. If there is no edge between nodes, $A_{ij} = 0$; otherwise, $A_{ij} > 0$ denotes an edge.

\[
L = D - A \\
\sum_j A_{ij}
\]

A signed graph can be encoded by its signed (weighted) adjacency matrix $W$. If there is no edge between nodes, $W_{ij} = 0$; otherwise, $W_{ij} > 0$ denotes a positive edge, while $W_{ij} < 0$ denotes a negative edge.

\[
L = D - W \\
\sum_j |W_{ij}|
\]

\[
\frac{dx}{dt} = -Lx
\]

2 nodes try to reach the same value (consensus)

2 nodes try to reach the same value (consensus)

2 nodes try to reach opposite values (dissensus)

Laplacian in unsigned and signed networks

The signed graph can be encoded by its signed adjacency matrix $A$. If there is no edge between nodes, $A_{ij} = 0$; otherwise, $A_{ij} > 0$ denotes an edge.

$$L = D - A$$

$$d_i = \sum_j A_{ij}$$

$$\lambda_1 = 0$$

$\lambda_2 = 0$ Graph is connected

The signed graph can be encoded by its signed (weighted) adjacency matrix $W$. If there is no edge between nodes, $W_{ij} = 0$; otherwise, $W_{ij} > 0$ denotes a positive edge, while $W_{ij} < 0$ denotes a negative edge.

$$L = D - W$$

$$d_i = \sum_j |W_{ij}|$$

$$\lambda_1 = 0$$

$\lambda_i \geq 0$ Graph is balanced

When the graph is balanced, the spectra of the signed and unsigned Laplacian can be mapped onto each other.
Graph is almost connected: communities are encoded in the (second) dominant eigenvectors (Fiedler, etc.,)

Graph is connected

Graph is almost balanced: communities are encoded in the dominant eigenvector

Graph is balanced

What if the network has more than 2 opposing groups?

A weaker condition for clusterability was proved by Davis, using the notion of weak balance to refer to graphs where **no cycle has only a single negative edge**. Graphs which exhibit weak balance, can be partitioned into **k clusters** with positive edges inside, and negative edges connecting them.

In the signed Laplacian, \(-\ (-) = +\)

**The enemy of my enemy is?**

Other Laplacians: SPONGE, Repelling Laplacian, etc.

J. A. Davis, Human relations 20, 181 (1967).
Repelling versus opposing Laplacian

$A_+ \text{ encodes the positive edges, and } A_- \text{ the negative edges}$

Quadratic form of the Laplacians ("energy")

$$x^T L_o x = \sum_{i,j} A_{ij}^+ |x_i - x_j|^2 + \sum_{i,j} A_{ij}^- |x_i + x_j|^2$$

Minimised when

$$x_i = -x_j$$

"one-dimensional bipolarisation"

$$x^T L_r x = \sum_{i,j} A_{ij}^+ |x_i - x_j|^2 - \sum_{i,j} A_{ij}^- |x_i - x_j|^2$$

Minimised when

$$|x_i - x_j| \rightarrow \infty$$

Allows for multiple polarisation in sufficiently many dimensions

FIG. 6: 3 Community SSBMs with 50 node communities and 0.5 edge probability, embedded using SHEEP (first two eigenvectors of repelling Laplacian). No edge sign flips (left) and 0.2 probability of sign flip (right).

11:45 Shazia Ayn Babul and Renaud Lambiotte
SHEEP: Signed Hamiltonian Eigenvector Embedding for Proximity
PRESENTER: Shazia Ayn Babul
Strong and weak random walks
Signed Laplacian and (strong) random walks

We consider the dynamics of two types of walkers, positive and negative walkers. Walkers perform a transition randomly but: if they take a negative edge, they change polarity.
Signed Laplacian and (strong) random walks

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\[
\begin{align*}
    n_{+;j}(t+1) &= \sum_i \left( n_{+;i}T_{+;ij} + n_{-;i}T_{-;ij} \right) \\
    n_{-;j}(t+1) &= \sum_i \left( n_{-;i}T_{+;ij} + n_{+;i}T_{-;ij} \right) \\
    T_{+;ij} &= A_{+;ij}/d_i \\
    T_{-;ij} &= A_{-;ij}/d_i
\end{align*}
\]

\(A_+\) encodes the positive edges, and \(A_-\) the negative edges.

The dynamics is a random walk on a “supra”-adjacency matrix

\[
A = \begin{pmatrix} A_+ & A_- \\ A_- & A_+ \end{pmatrix} \quad T = \begin{pmatrix} T_+ & T_- \\ T_- & T_+ \end{pmatrix}
\]

Yu Tian and Renaud Lambiotte
59. Spreading and Structural Balance on Signed Networks
PRESENTER: Yu Tian
Signed Laplacian and (strong) random walks

We consider the dynamics of two types of walkers, positive and negative walkers. Walkers perform a transition randomly but: if they take a negative edge, they change polarity.

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n_{+;j}(t+1) &= \sum_i \left(n_{+;i}T_{+;ij} + n_{-;i}T_{-;ij}\right) \\
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\end{align*}
\]

\(A_+\) encodes the positive edges, and \(A_-\) the negative edges

The dynamics is a random walk on a “supra”-adjacency matrix

\[
\Delta_i = n_{+;i} - n_{-;i} \\
n_i = n_{+;i} + n_{-;i}
\]

\[
T' = \begin{pmatrix} T_s & 0 \\ 0 & T_u \end{pmatrix}
\]

\(T\) -> Unsigned and signed (normalised) Laplacian (finds bottlenecks and polarisation)
Structural Balance and Random Walks on Complex Networks with Complex Weights

Yu Tian, Renaud Lambiotte

Complex numbers define the relationship between entities in many situations. A canonical example would be the off-diagonal terms in a Hamiltonian matrix in quantum physics. Recent years have seen an increasing interest to extend the tools of network science when the weight of edges are complex numbers. Here, we focus on the case when the weight matrix is Hermitian, a reasonable assumption in many applications, and investigate both structural and dynamical properties of the complex-weighted networks. Building on concepts from signed graphs, we introduce a classification of complex-weighted networks based on the notion of structural balance, and illustrate the shared spectral properties within each type. We then apply the results to characterise the dynamics of random walks on complex-weighted networks, where local consensus can be achieved asymptotically when the graph is structurally balanced, while global consensus will be obtained when it is strictly unbalanced. Finally, we explore potential applications of our findings by generalising the notion of cut, and propose an associated spectral clustering algorithm. We also provide further characteristics of the magnetic Laplacian, associating directed networks to complex-weighted ones. The performance of the algorithm is verified on both synthetic and real networks.
Kernels, problems and how to solve them

Unsigned networks

Real-world networks are sparse, and only a small fraction of the pairs of nodes are connected. Random-walk-based kernels allow to estimate the proximity of pairs of nodes. In case of unsigned networks, the resulting similarity between two nodes is obtained from an appropriately weighted sum of the walks between them. Typically, the existence of many short walks between two nodes ensures their proximity. E.g. Heat kernel

Signed networks

Based on the strong balance: Two nodes are similar if there exist many, short positive walks (even number of negative edges), and are dissimilar if there exist many, short negative walks (odd number of negative edges). Kernels are directly derived from strong random walks.
Kernels, problems and how to solve them

Unsigned networks

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Signed networks

Based on the strong balance: Two nodes are similar if there exist many, short positive walks (even number of negative edges), and are dissimilar if there exist many, short negative walks (odd number of negative edges). Kernels are directly derived from strong random walks.

Perfectly clustered, yet contradictions for the walk
Kernels, problems and how to solve them

**Strong walks**

Based on the strong balance:
Two nodes are similar if there exist many, short positive walks (even number of negative edges), and are dissimilar if there exist many, short negative walks (odd number of negative edges).
Kernels are directly derived from strong random walks.

\[
T = \begin{pmatrix}
T_+ & T_- \\
T_- & T_+
\end{pmatrix}
\]

**Weak walks**

Based on the weak balance:
Two nodes are similar if there exist many, short positive walks (made only of positive edges), and are dissimilar if there exist many, short negative walks (walks with one single negative edge).
Any other edges (with at least 2 negative edges) is not informative
Kernels are directly derived from strong random walks.

\[
T = \begin{pmatrix}
T_+ & T_- \\
0 & T_+
\end{pmatrix}
\]

Enemy of my enemy is?
Kernels, problems and how to solve them

S. Babul, J. Pougué Biyong, Z. Schwerkolt and R. Lambiotte, in preparation
... and more

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Strong, weak or no balance? Testing structural hypotheses in real heterogeneous networks

Anna Gallo, Diego Garlaschelli, Renaud Lambiotte, Fabio Saracco, Tiziano Squartini
Many thanks to my collaborators and students, especially Yu Tian, Shazia Babul and John Pougué Biyong for this project.
Call for Papers

Structure and Dynamics of Signed Networks

Guest Editors

Renaud Lambiotte (University of Oxford)
Vincent Traag (Leiden University)