Dynamics and Balance on Signed Networks

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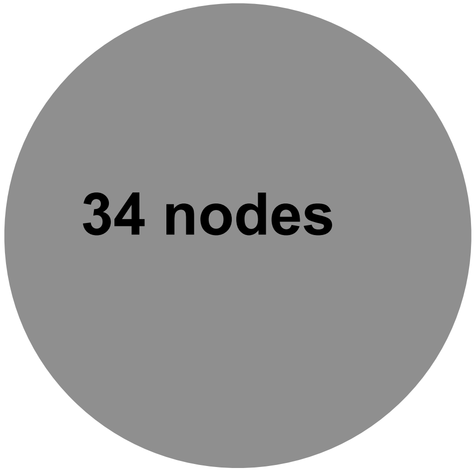
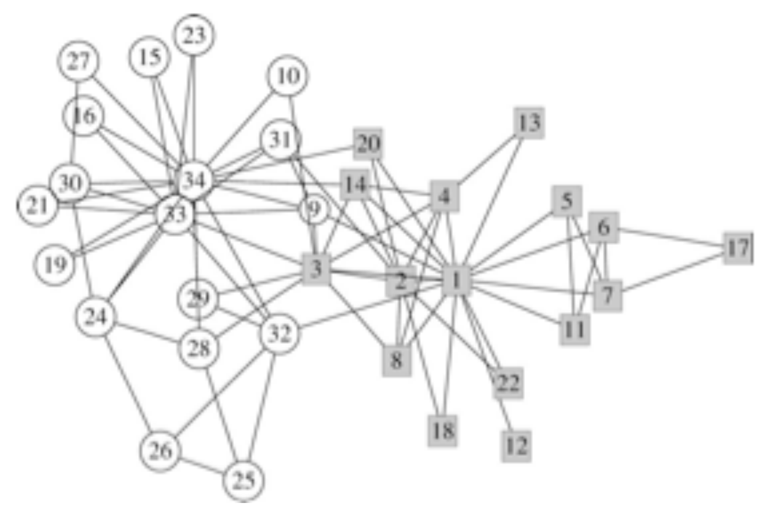
A network science approach to signed graphs

Change of perspective



Compared with mean-field approaches, which summarise interactions between all elements with a single **averaged** field, network models often have much higher explanatory power because they can account for **sparse** and **non-random** topologies.

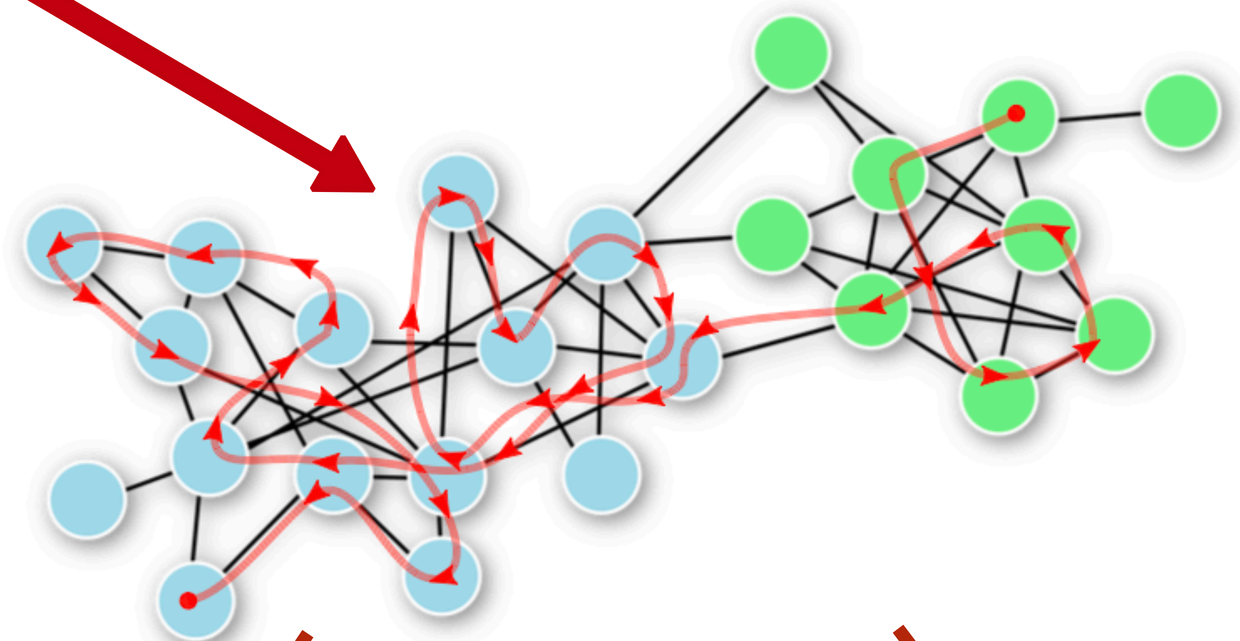
To capture **direct** interactions, a network represents components with nodes
and pairwise interactions.
To capture **indirect** interactions, one defines the notion of walk/path/connectivity



Data, Dynamics and structure

DATA

Fuel for the theoretical modelling



Effect of topology on spreading:
What are the network properties that slow down or accelerate the dynamics?

Uncover structure from dynamics:
What are the important nodes or the important substructures in the graph?
Community detection, graph embeddings, etc.

Data collection and community detection

Empirical data in social media?

DEBAGREEMENT: A comment-reply dataset for (dis)agreement detection in online debates

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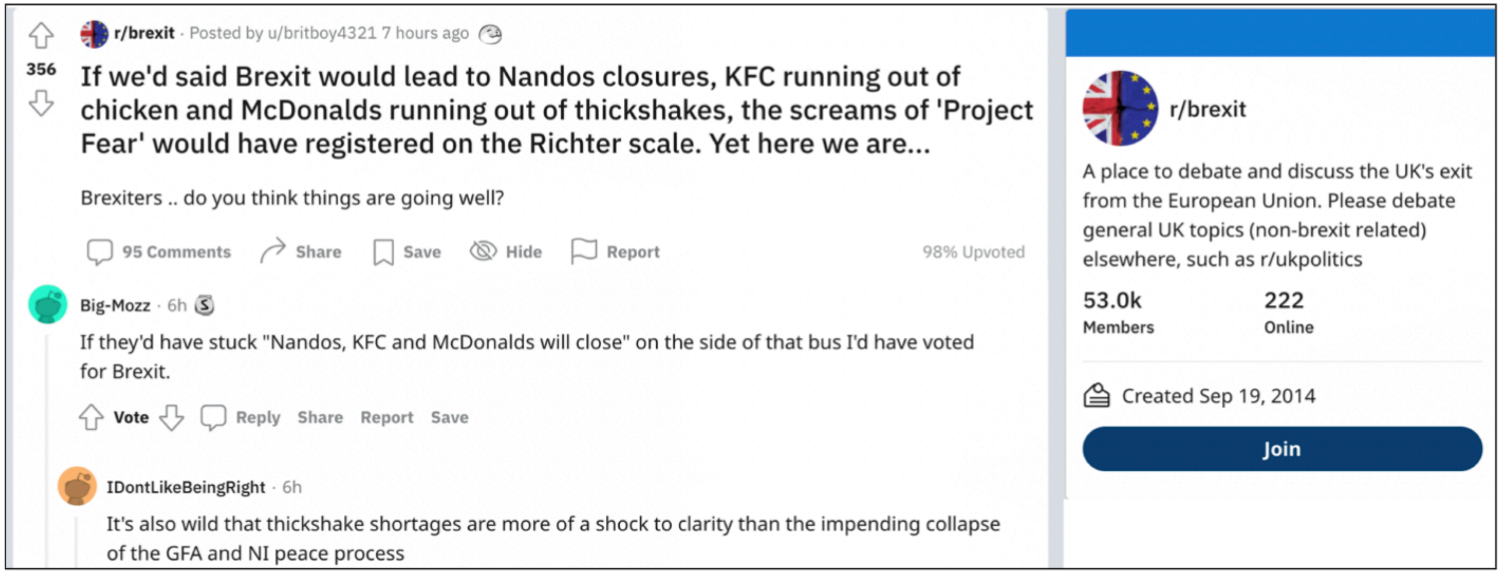


Figure 1: `r/Brexit`

- `r/BlackLivesMatter` discusses news related to the *Black Lives Matter* movement. It was created in 2014 and has 109K members,
- `r/Brexit` aims to foster debate about the United Kingdom's (UK) exit from the European Union (EU). It was created in 2014 and has 53K members,
- `r/climate` is a community for truthful science-based news about climate and related politics and activism. It was created in 2008 and has 99K members,
- `r/democrats` is a partisan subreddit. It aims to discuss political news, policies and how to ensure the election of Democratic party candidates. It was created in 2014 and has 292K members,
- `r/Republican` is a partisan subreddit for Republicans to discuss issues with each other. It was created in 2008 and has 172K members.

Graph creation For each subreddit `r/*`, the resulting set of interactions forms a multi-edge, temporal graph $\mathcal{G}_{r/*}$, where nodes are users, and edges represent a comment-reply interaction between two users. One of the unique advantages of DEBAGREEMENT over other datasets is the additional graph interaction information provided about every subreddit.

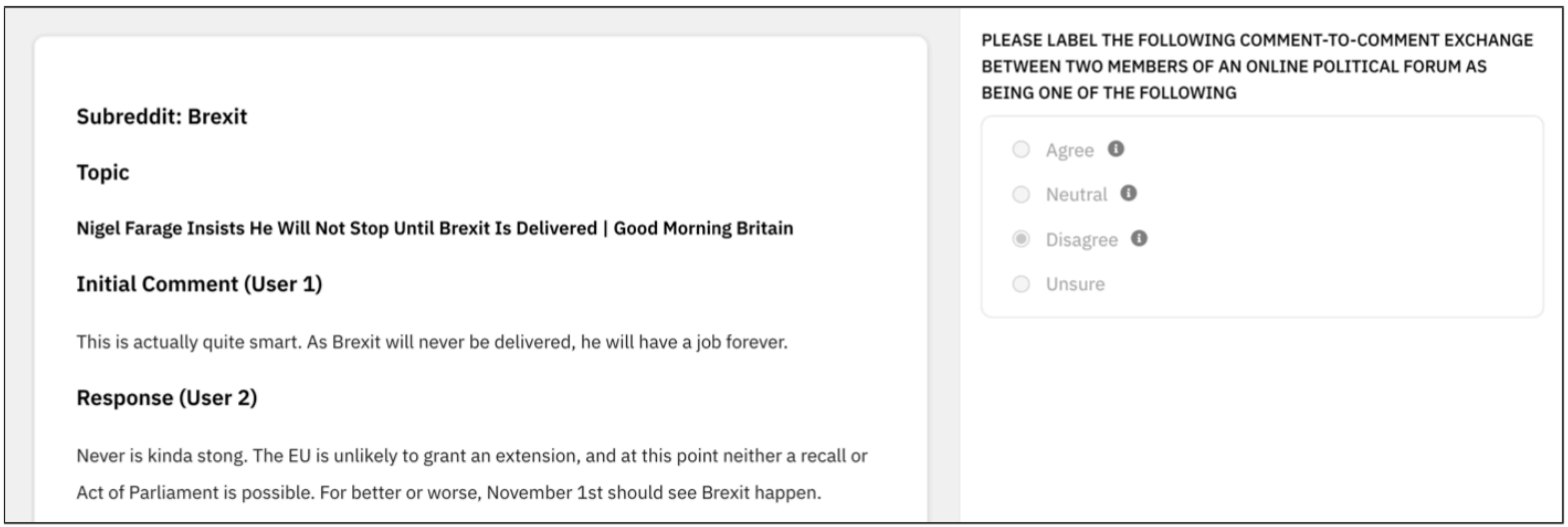


Figure 2: User interface for annotators

Table 1: Dataset statistics

	r/Brexit	r/climate	r/BLM	r/Republican	r/democrats
Start date	Jun 2016	Jan 2015	Jan 2020	Jan 2020	Jan 2020
#nodes	722	4,580	2,516	8,832	6,925
#edges	15,745	5,773	1,929	9,823	9,624
positive	29%	32%	45%	34%	42%
neutral	29%	28%	22%	25%	22%
negative	42%	40%	33%	41%	36%

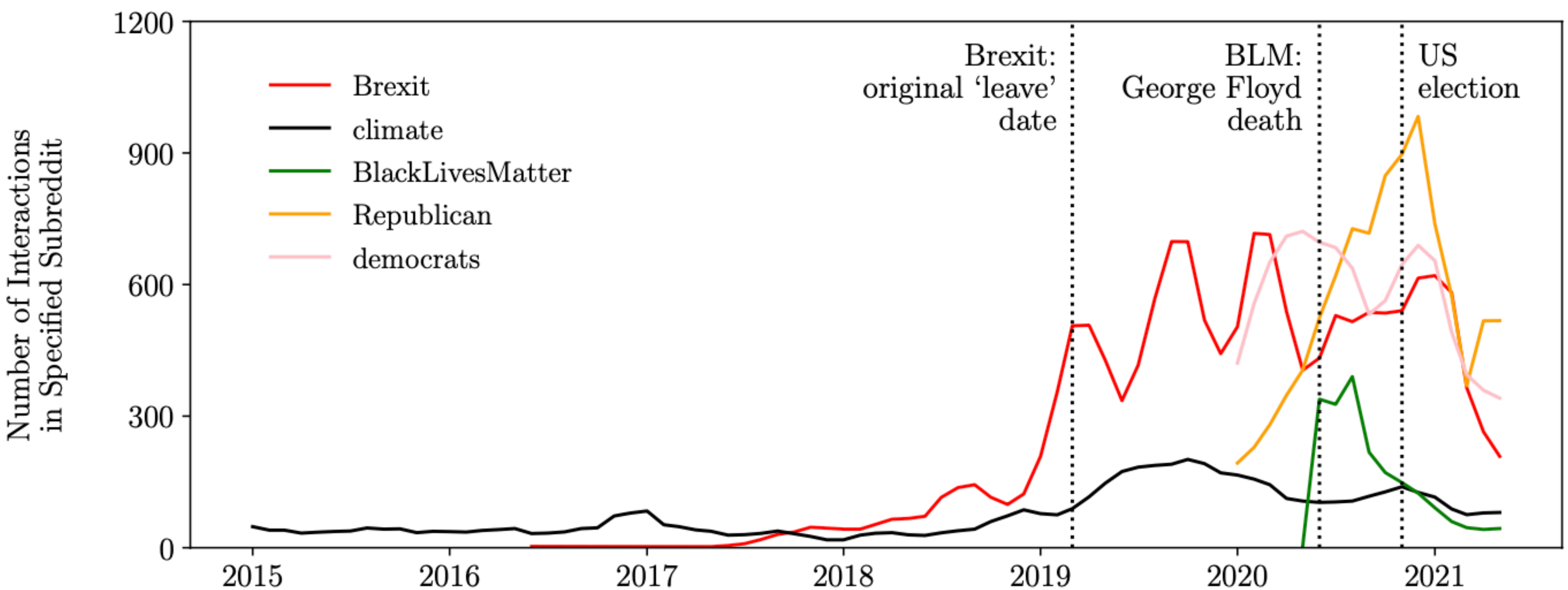
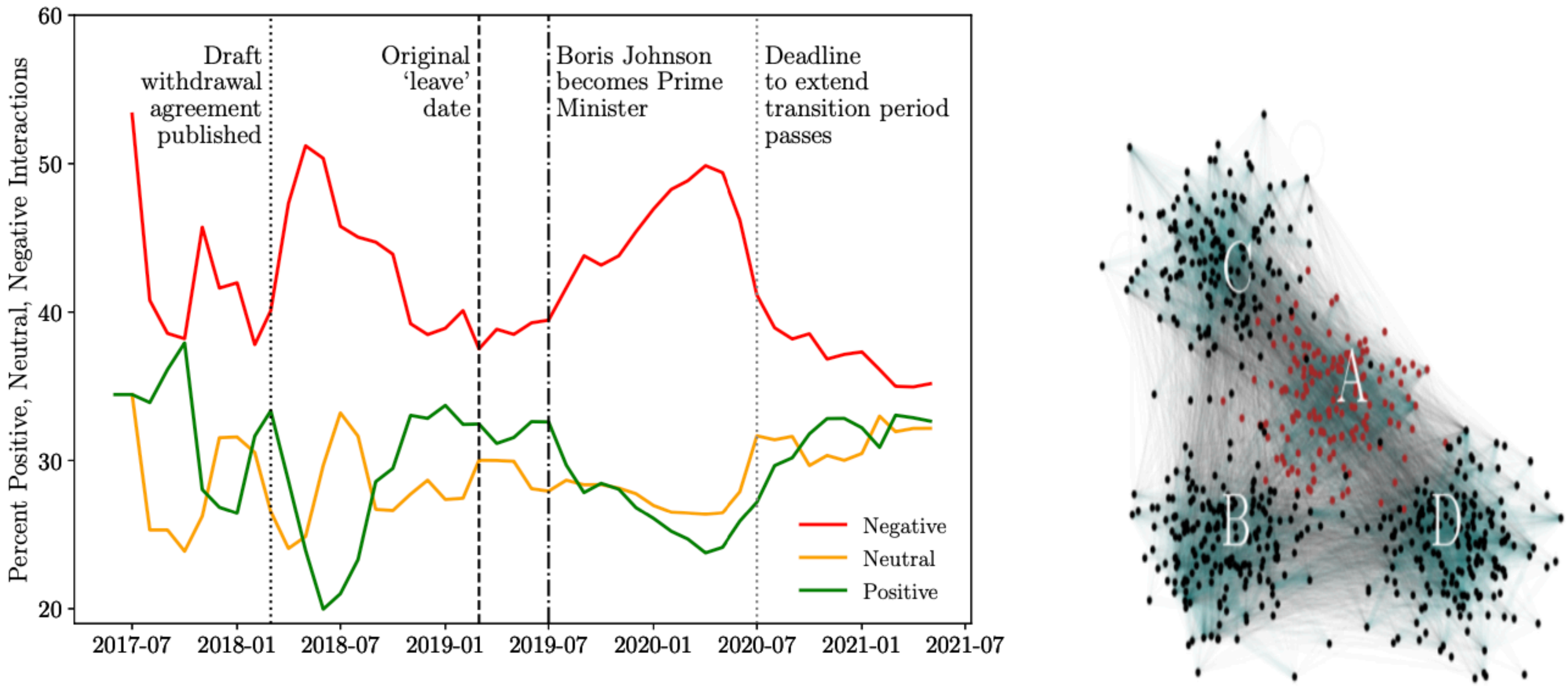


Figure 3: Number of interactions per subreddit (3-month rolling averages)



(a) Interactions in `r/Brexit` (3-month rolling averages) (b) Communities in `r/Brexit`

Figure 4: Polarisation in `r/Brexit` over time

most active users of each community, we conclude that community **A** (in brown) is pro-Brexit and communities **B**, **C** and **D** (in black) express sentiments in favor of the UK remaining in the EU. We look further at the main topics of discussion in each community and conclude that: users in community **B** are interested in the consequences of Brexit on international trade, users in community **C** discuss the accountability of UK political figures in what they consider 'a disaster' for the UK, and users in community **D** are mostly interested in UK-EU negotiations and the votes in UK parliament.

Optimising signed modularity

To cluster signed networks, the purpose is to place negative edges between communities and positive edges inside communities.

An efficient way based on NG modularity is to consider the signed network as the combination of two networks, defined by the positive and negative edges respectively, and to seek to optimise the difference of their modularities.

Positive edges inside

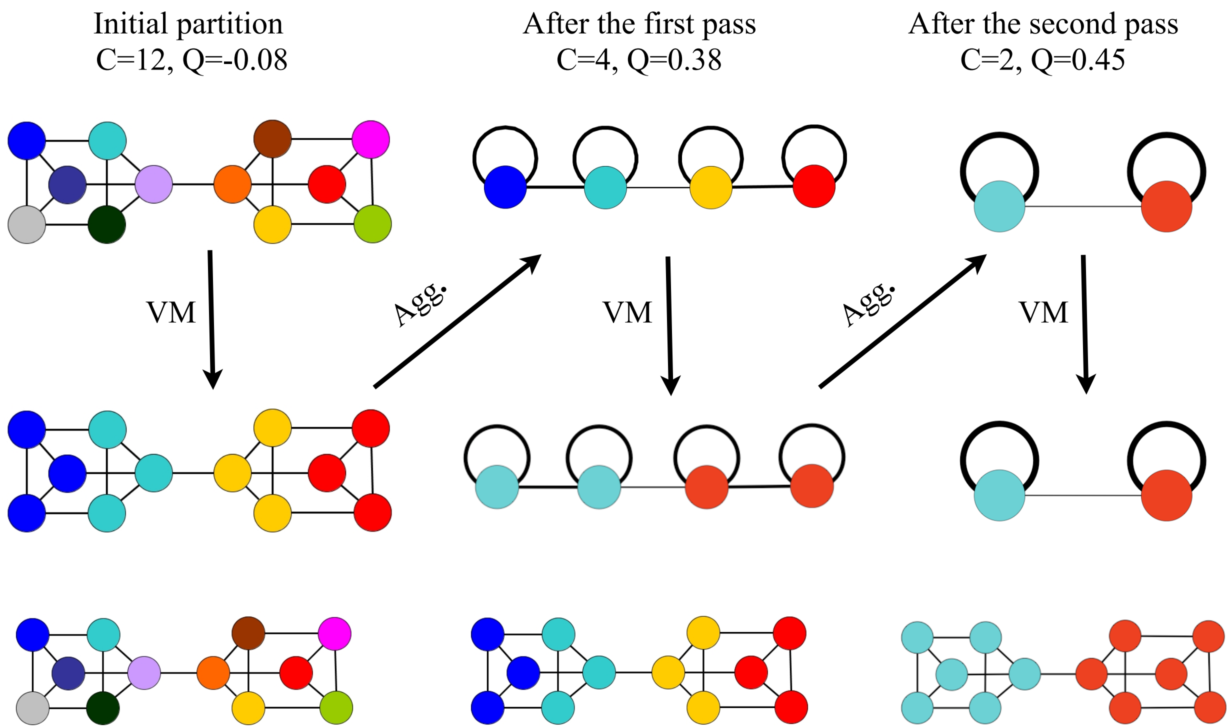
Negative edges outside

$$Q = Q^+ - Q^-$$

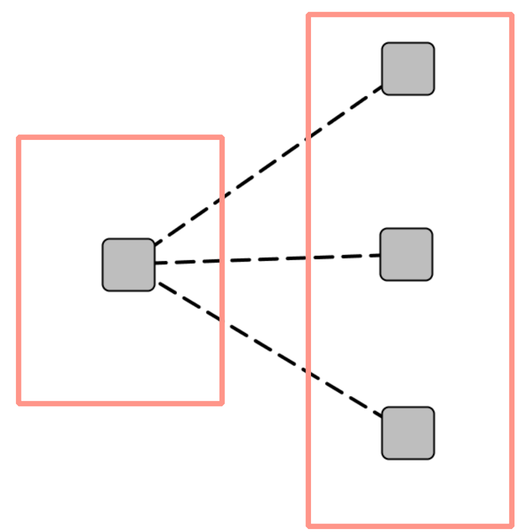
$$Q = \sum_{ij} \left[A_{ij} - \left(\frac{d_i^+ d_j^+}{2m^+} - \frac{d_i^- d_j^-}{2m^-} \right) \right] \delta(\sigma_i, \sigma_j)$$

V. A. Traag and J. Bruggeman, Physical Review E 80, 036115 (2009).

Louvain for signed networks



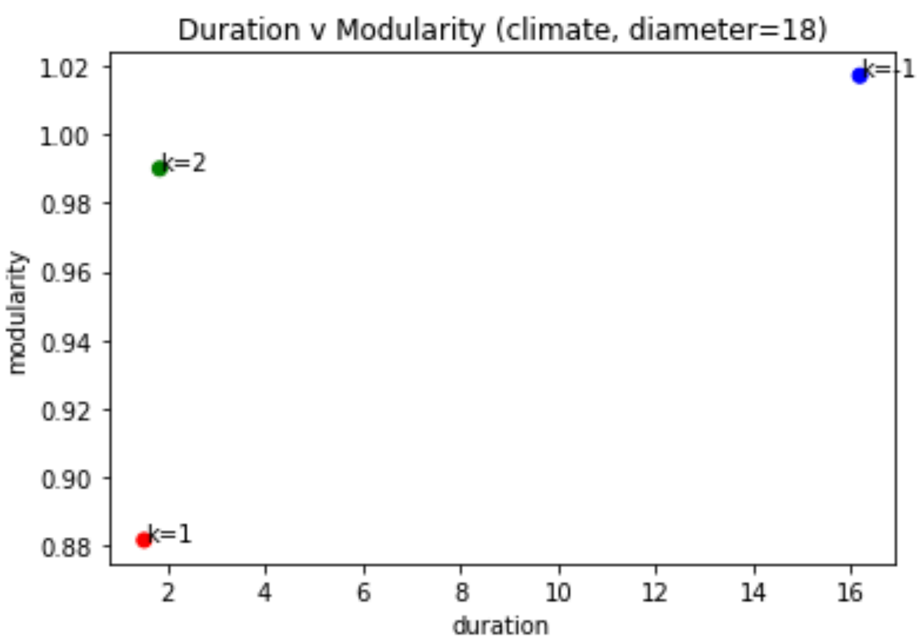
In the VM step, one tries to move a node to the community of its neighbours: sensible choice for positive relations, as nodes should be in the community of one of their neighbours, but not for negative relations, as nodes should actually be placed in a community different from that of their negative connections.

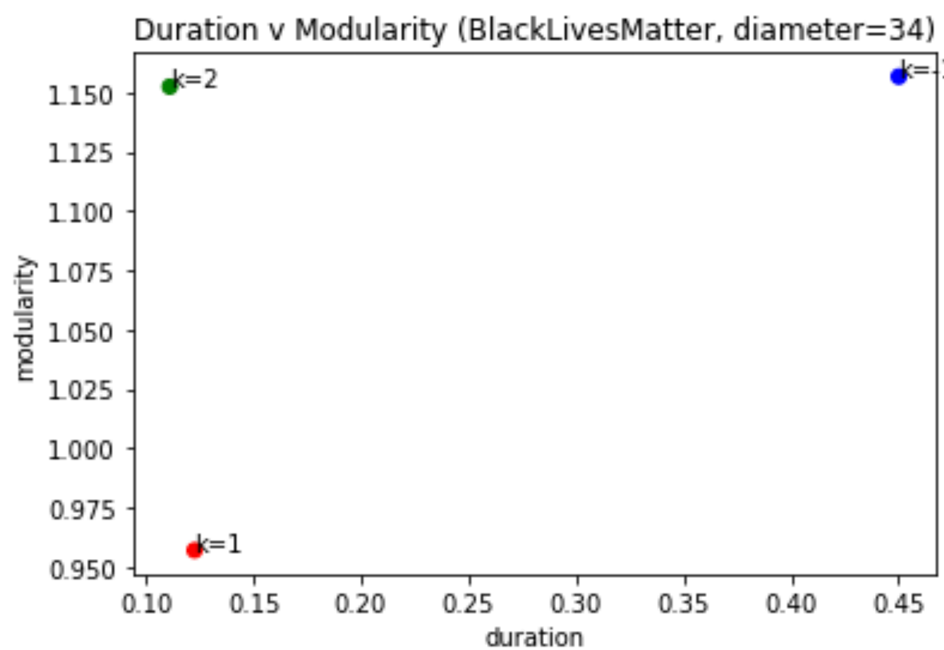


Louvain for signed networks

In the VM step, one could try to move tries to move a node to the any community, not only of the neighbours. Works very well but slows down the method drastically (no locality in the optimisation).

Alternative is to try to move a node to the community of its first neighbour for positive edges, and second neighbour for negative edges (“the enemy of my enemy *could* be my friend”).





https: //pypi.python.org/pypi/louvain/
J. Pougué Biyong and R. Lambiotte, in preparation.

Laplacian for signed networks

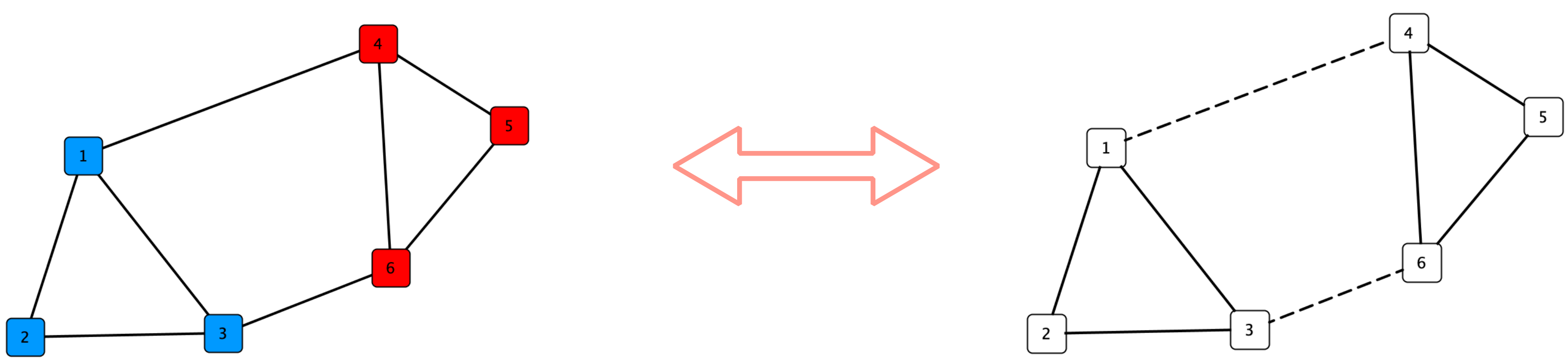
Important structures in a signed graph?

Introduced in 1940s and primarily motivated by social and economic networks, a fundamental notion in the study of signed networks is the so-called **structural balance**.

A signed graph is structurally balanced if and only if there is no cycle with an odd number of negative edges, which defines the cycle to be “negative”.

The following theorem provides an alternative interpretation of structural balance in terms of a bipartition of signed graphs.

Theorem 2.1 (Structure Theorem for Balance [22]). *A signed graph G is structurally balanced if and only if there is a bipartition of the node set into $V = V_1 \cup V_2$ with V_1 and V_2 being mutually disjoint and one of them being nonempty, s.t. any edge between the two node subsets is negative while any edge within each node subset is positive.*



F. Heider. Attitudes and cognitive organization. J. Psychol., 21(1):107–112, 1946.
D. Cartwright and F. Harary. Structural balance: A generalization of heider’s theory. Psychol. Rev., 63(5):277–293, 1956.

Laplacian in unsigned and signed networks

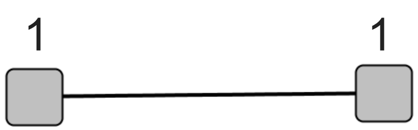
An unsigned graph can be encoded by its signed adjacency matrix **A**.
If there is no edge between nodes, $A_{ij} = 0$; otherwise, $A_{ij} > 0$ denotes an edge.

$$\mathbf{L} = \mathbf{D} - \mathbf{A} \qquad d_i = \sum_j A_{ij}$$

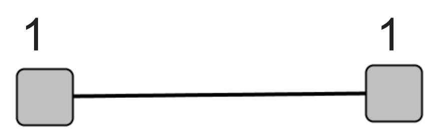
A signed graph can be encoded by its signed (weighted) adjacency matrix **W**.
If there is no edge between nodes, $W_{ij} = 0$; otherwise, $W_{ij} > 0$ denotes a positive edge, while $W_{ij} < 0$ denotes a negative edge.

$$\mathbf{L} = \mathbf{D} - \mathbf{W} \qquad d_i = \sum_j |W_{ij}|$$

$$\frac{d}{dt}\mathbf{x} = -\mathbf{L}\mathbf{x}$$



2 nodes try to reach the same value (consensus)



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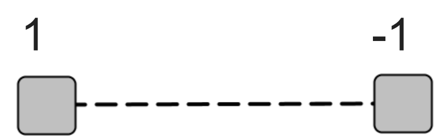


2 nodes try to reach opposite values (dissensus)

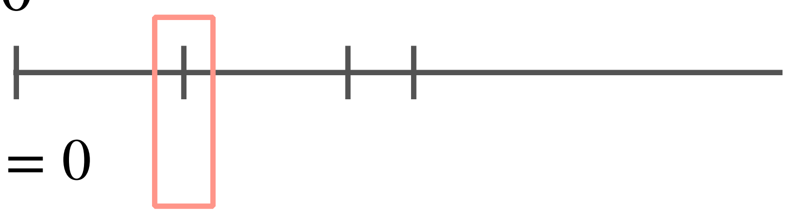
J. Kunegis, S. Schmidt, A. Lommatzsch, J. Lerner, E. De Luca, and S. Albayrak. Spectral analysis of signed graphs for clustering, prediction and visualization. In Proceedings of the 2010 SIAM International Conference on Data Mining, pages 559–570. SIAM, 2010.

Laplacian in unsigned and signed networks

The signed graph can be encoded by its signed adjacency matrix **A**.
If there is no edge between nodes, $A_{ij} = 0$; otherwise, $A_{ij} > 0$ denotes an edge.

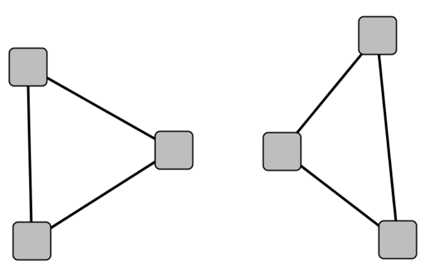
$$\mathbf{L} = \mathbf{D} - \mathbf{A} \qquad d_i = \sum_j A_{ij}$$

0



$\lambda_1 = 0$

$\lambda_2 = 0$ Graph is connected



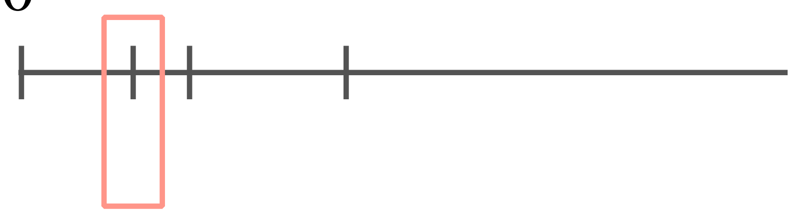
$$\mathbf{L}_{rw} = \mathbf{I} - \mathbf{D}^{-1}\mathbf{W}$$

$\lambda_i \geq 0$

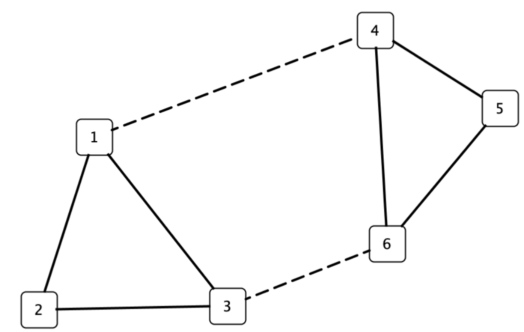
The signed graph can be encoded by its signed (weighted) adjacency matrix **W**.
If there is no edge between nodes, $W_{ij} = 0$; otherwise, $W_{ij} > 0$ denotes a positive edge, while $W_{ij} < 0$ denotes a negative edge.

$$\mathbf{L} = \mathbf{D} - \mathbf{W} \qquad d_i = \sum_j |W_{ij}|$$

0



$\lambda_1 = 0$ Graph is balanced



When the graph is balanced, the spectra of the signed and unsigned Laplacian can be mapped onto each other.

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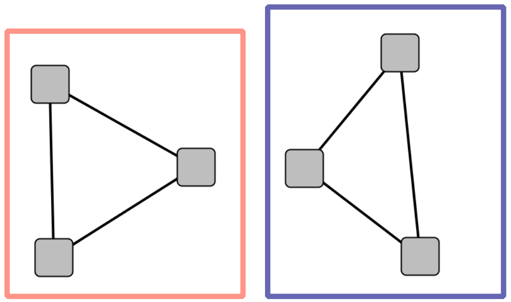
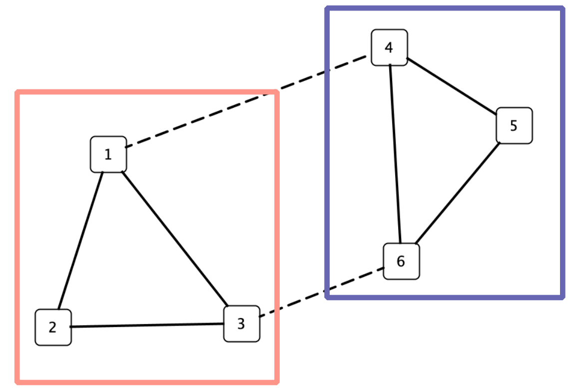
59. Spreading and Structural Balance on Signed Networks
PRESENTER: Yu Tian

Spectral methods

Graph is connected



Graph is balanced



Graph is almost connected: communities are encoded in the (second) dominant eigenvectors (Fiedler, etc.,)

Graph is almost balanced: communities are encoded in the dominant eigenvector

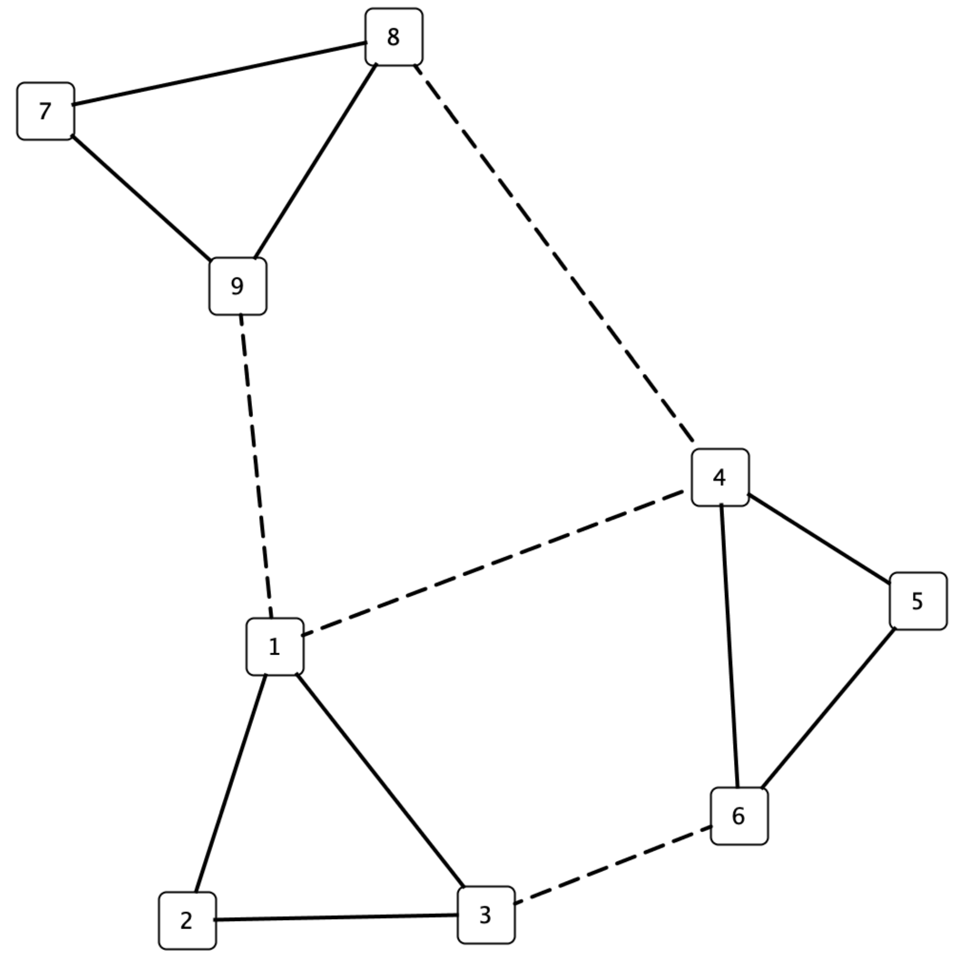
J. Kunegis, S. Schmidt, A. Lommatzsch, J. Lerner, E. De Luca, and S. Albayrak. Spectral analysis of signed graphs for clustering, prediction and visualization. In Proceedings of the 2010 SIAM International Conference on Data Mining, pages 559–570. SIAM, 2010.

What if the network has more than 2 opposing groups?

A weaker condition for clusterability was proved by Davis, using the notion of weak balance to refer to graphs where **no cycle has only a single negative edge**. Graphs which exhibit weak balance, can be partitioned into **k clusters** with positive edges inside, and negative edges connecting them.

In the signed Laplacian, - (-) = +
The enemy of my enemy is?

Other Laplacians: SPONGE, Repelling Laplacian, etc.



J. A. Davis, Human relations 20, 181 (1967).
M. Cucuringu, P. Davies, A. Glielmo, and H. Tyagi, in The 22nd International Conference on Artificial Intelligence and Statistics (PMLR, 2019) pp. 1088–1098.
A. Knyazev, in 2018 Proceedings of the Seventh SIAM Workshop on Combinatorial Scientific Computing (SIAM, 2018) pp. 11–22.

Repelling versus opposing Laplacian

A_+ encodes the positive edges, and A_- the negative edges
Quadratic form of the Laplacians ("energy")

$$\mathsf{x}^T L_o \mathsf{x} = \sum_{i,j} A^+_{ij} |x_i - x_j|^2 + \boxed{\sum_{i,j} A^-_{ij} |x_i + x_j|^2}$$

Minimised when

$$x_i = -x_j$$

"one-dimensional bipolarisation"

$$\mathsf{x}^T L_r \mathsf{x} = \sum_{i,j} A^+_{ij} |x_i - x_j|^2 - \boxed{\sum_{i,j} A^-_{ij} |x_i - x_j|^2}$$

Minimised when

$$|x_i - x_j| \to \infty$$

Allows for multiple polarisation in sufficiently many dimensions

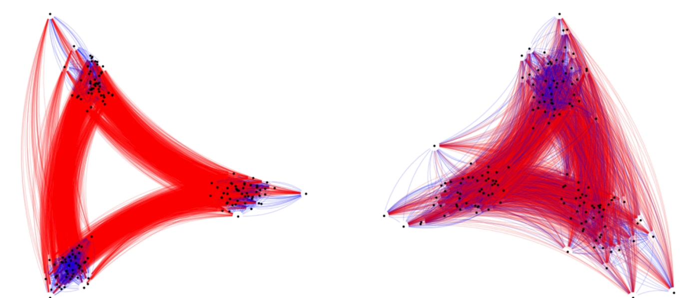


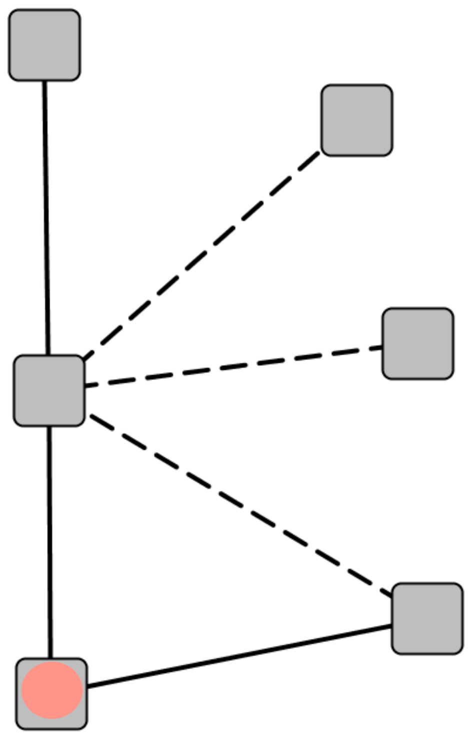
FIG. 6: 3 Community SSBMs with 50 node communities and 0.5 edge probability, embedded using SHEEP (first two eigenvectors of repelling Laplacian). No edge sign flips (left) and 0.2 probability of sign flip (right).

11:45 Shazia Ayn Babul and Renaud Lambiotte
SHEEP: Signed Hamiltonian Eigenvector Embedding for Proximity
PRESENTER: Shazia Ayn Babul

Strong and weak random walks

Signed Laplacian and (strong) random walks

We consider the dynamics of two types of walkers, positive and negative walkers. Walkers perform a transition randomly but: if they take a negative edge, they change polarity.



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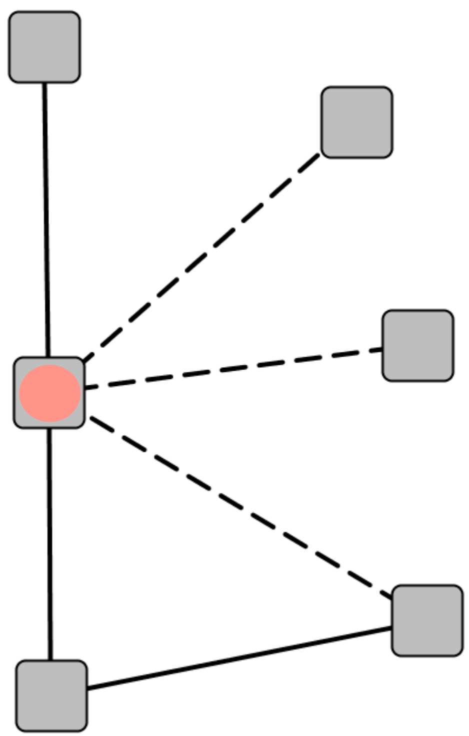

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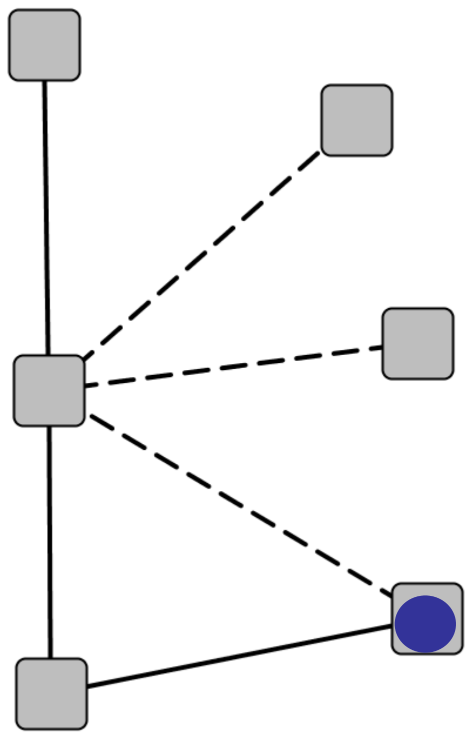

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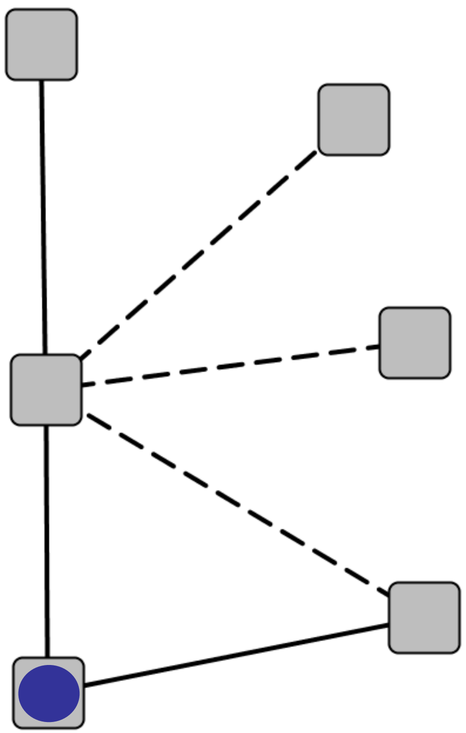

$$n_{+;j}(t+1) = \sum_i \left(n_{+;i} T_{+;ij} + n_{-;i} T_{-;ij} \right)$$

$$n_{-;j}(t+1) = \sum_i \left(n_{-;i} T_{+;ij} + n_{+;i} T_{-;ij} \right)$$

$$T_{+;ij} = A_{+;ij}/d_i$$

$$T_{-;ij} = A_{-;ij}/d_i$$

A_+ encodes the positive edges, and A_- the negative edges

The dynamics is a random walk on a "supra"-adjacency matrix

$$A = \begin{pmatrix} A_+ & A_- \\ A_- & A_+ \end{pmatrix} \qquad T = \begin{pmatrix} T_+ & T_- \\ T_- & T_+ \end{pmatrix}$$

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PRESENTER: Yu Tian

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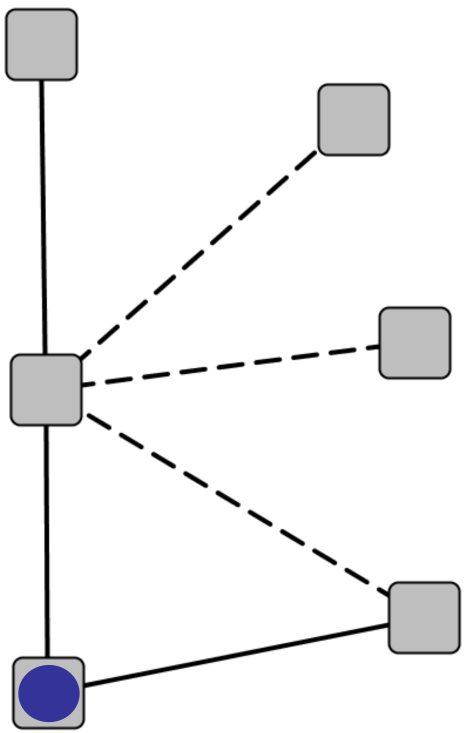

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$$n_{-;j}(t+1) = \sum_i \left(n_{-;i} T_{+;ij} + n_{+;i} T_{-;ij} \right)$$

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A_+ encodes the positive edges, and A_- the negative edges

The dynamics is a random walk on a “supra”-adjacency matrix

$$\Delta_i = n_{+;i} - n_{-;i}$$
$$n_i = n_{+;i} + n_{-;i}$$

$$T' = \begin{pmatrix} T_s & 0 \\ 0 & T_u \end{pmatrix}$$

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PRESENTER: Yu Tian

-> Unsigned and signed (normalised) Laplacian (finds bottlenecks and polarisation)

From signed networks to complex-weighted networks

[Submitted on 4 Jul 2023]

Structural Balance and Random Walks on Complex Networks with Complex Weights

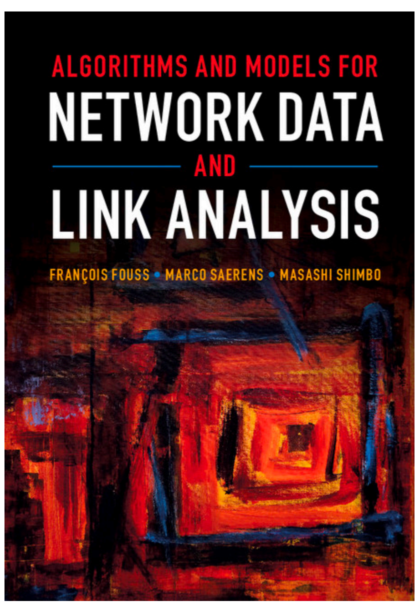
Yu Tian, Renaud Lambiotte

Complex numbers define the relationship between entities in many situations. A canonical example would be the off-diagonal terms in a Hamiltonian matrix in quantum physics. Recent years have seen an increasing interest to extend the tools of network science when the weight of edges are complex numbers. Here, we focus on the case when the weight matrix is Hermitian, a reasonable assumption in many applications, and investigate both structural and dynamical properties of the complex-weighted networks. Building on concepts from signed graphs, we introduce a classification of complex-weighted networks based on the notion of structural balance, and illustrate the shared spectral properties within each type. We then apply the results to characterise the dynamics of random walks on complex-weighted networks, where local consensus can be achieved asymptotically when the graph is structurally balanced, while global consensus will be obtained when it is strictly unbalanced. Finally, we explore potential applications of our findings by generalising the notion of cut, and propose an associated spectral clustering algorithm. We also provide further characteristics of the magnetic Laplacian, associating directed networks to complex-weighted ones. The performance of the algorithm is verified on both synthetic and real networks.

Kernels, problems and how to solve them

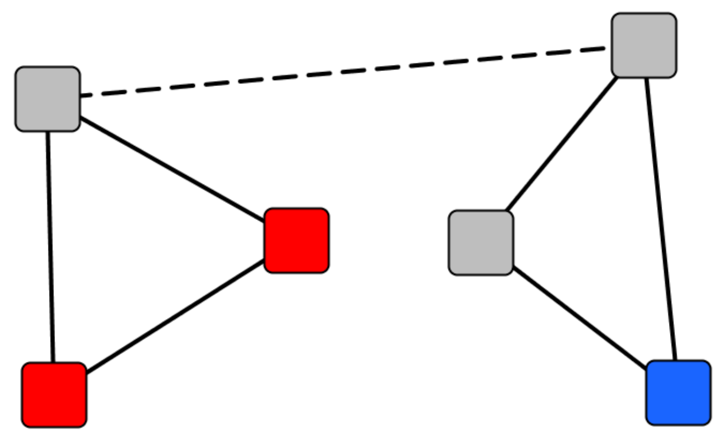
Unsigned networks

Real-world networks are sparse, and only a small fraction of the pairs of nodes are connected.
Random-walk-based kernels allow to estimate the proximity of pairs of nodes.
In case of unsigned networks, the resulting similarity between two nodes is obtained from an appropriately weighted sum of the walks between them.
Typically, **the existence of many short walks between two nodes ensures their proximity**. E.g. Heat kernel



Signed networks

Based on the strong balance:
Two nodes are **similar** if there exist **many, short positive walks** (even number of negative edges), and are **dissimilar** if there exist **many, short negative walks** (odd number of negative edges).
Kernels are directly derived from strong random walks.



Kernels, problems and how to solve them

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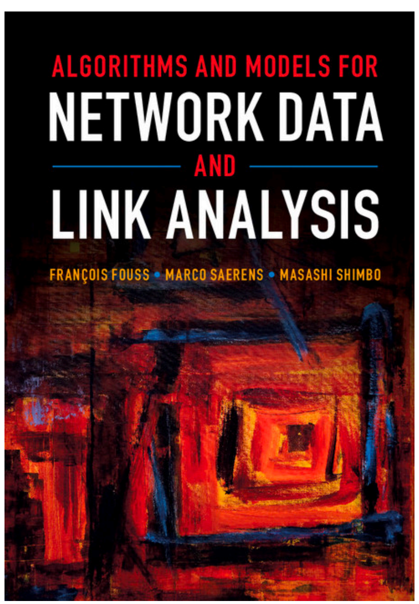

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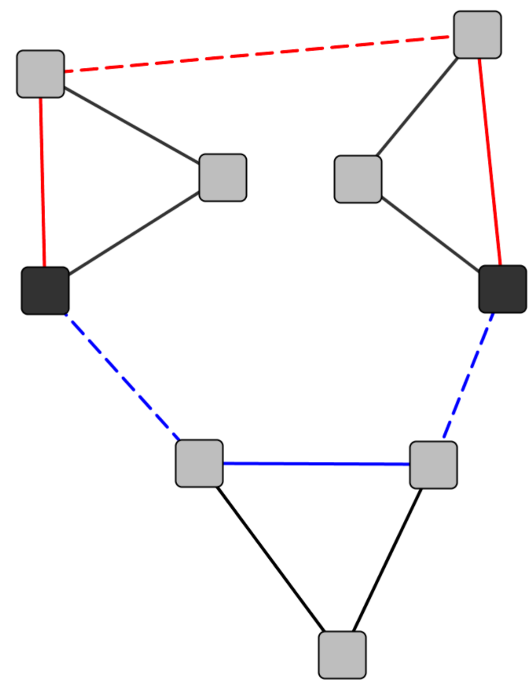


Perfectly clustered, yet contradictions for the walk

Kernels, problems and how to solve them

Strong walks

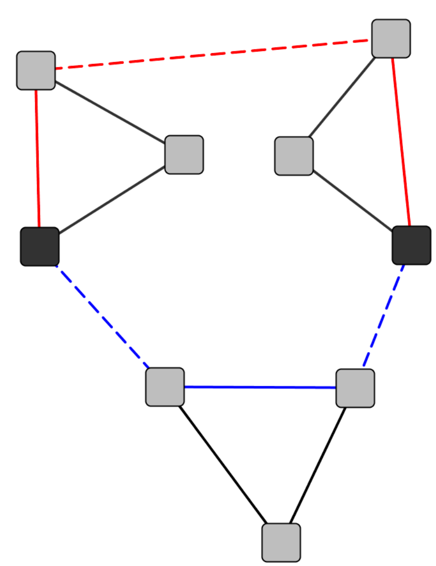
Based on the strong balance:
Two nodes are similar if there exist many, short positive walks (even number of negative edges), and are dissimilar if there exist many, short negative walks (odd number of negative edges).
Kernels are directly derived from strong random walks.

$$T = \begin{pmatrix} T_+ & T_- \\ T_- & T_+ \end{pmatrix}$$

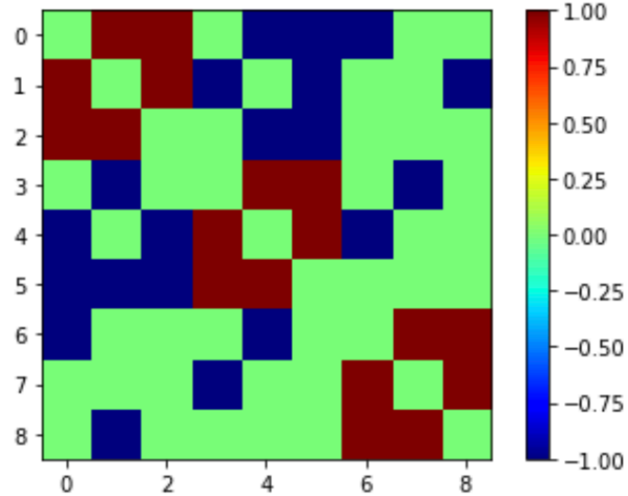
Weak walks

Based on the weak balance:
Two nodes are similar if there exist many, short positive walks (made only of positive edges), and are dissimilar if there exist many, short negative walks (walks with one single negative edge).
Any other edges (with **at least 2 negative edges**) is **not informative**
Kernels are directly derived from strong random walks.
Enemy of my enemy is?

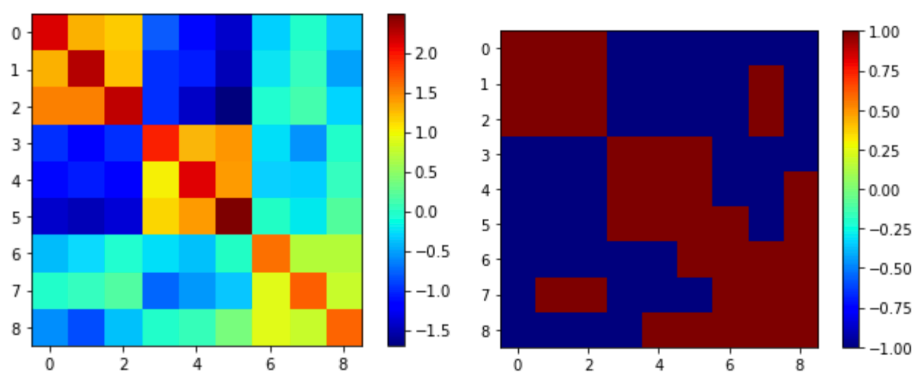
$$T = \begin{pmatrix} T_+ & T_- \\ 0 & T_+ \end{pmatrix}$$



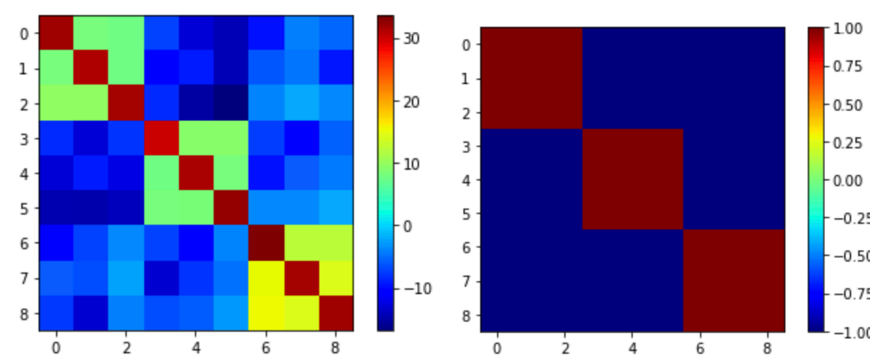
Kernels, problems and how to solve them



Strong walks

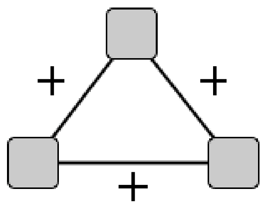
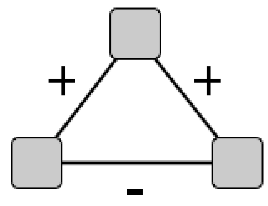
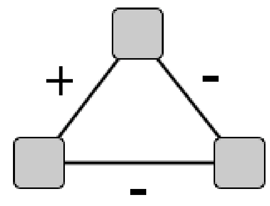
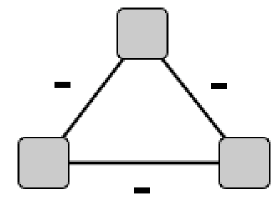


Weak walks



S. Babul, J. Pougué Biyong, Z. Schwerkolt and R. Lambiotte, in preparation

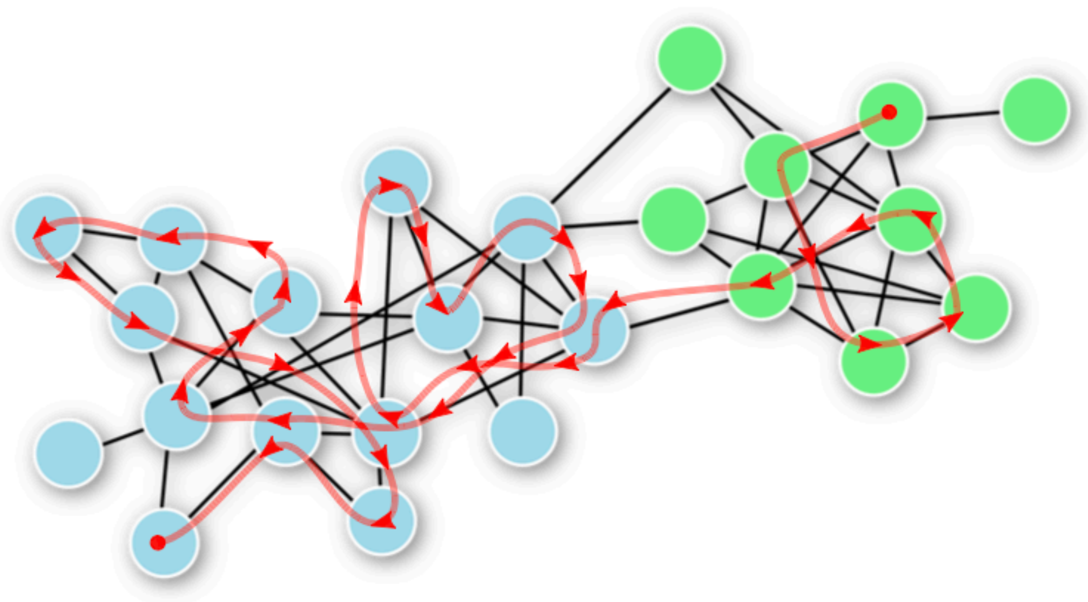
… and more

				
Strong formulation of balance	B	U	B	U
Weak formulation of balance	B	U	B	B
N_{Δ}	26,329	4,428	39,519	8,032
$N_{\Delta,r}$	10,608	30,145	28,545	9,009
z	71	-112	47	-5

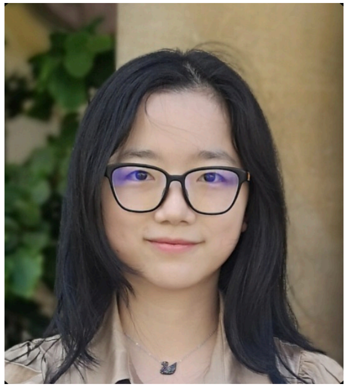
Strong, weak or no balance? Testing structural hypotheses in real heterogeneous networks

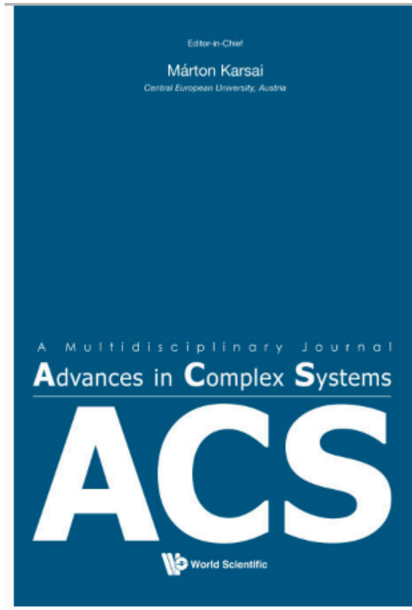
Anna Gallo, Diego Garlaschelli, Renaud Lambiotte, Fabio Saracco, Tiziano Squartini

Conclusion



Many thanks to my collaborators and students, especially Yu Tian, Shazia Babul and John Pougué Biyong for this project.





Advances in Complex Systems

A Multidisciplinary Journal

ISSN (print): 0219-5259 | ISSN (online): 1793-6802

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