Università degli Studi di Napoli
Federico II

Partial control of complex networks and its application to opinion dynamics

by

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OUTLINE

- ❑ Network Control
- ❑ Node Controllability and Observability
- ❑ Partial Pinning Control
- ❑ Modeling influencers as pinners in social systems

Network Control

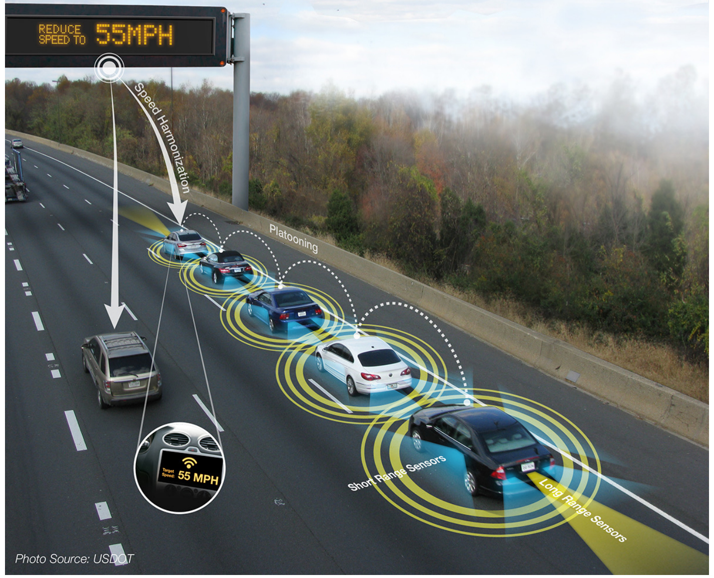


Photo Source: USDOT

- ❑ Converging towards some collective behavior
- ❑ Controlling the trajectory of the achieved collective behavior
- ❑ Reaching a predefined point of the state space

Control of collective behavior

$$\dot{x}_i = f(x_i) + c\sum_j a_{ij}(h(x_j) - h(x_i))$$

We know

1. when and how it is possible to ensure that [1,2]

$$\lim_{t\to\infty}\left|x_i(t) - x_j(t)\right| = 0 \ \forall i,j$$

2. where and how to inject input signals so that [3,4]

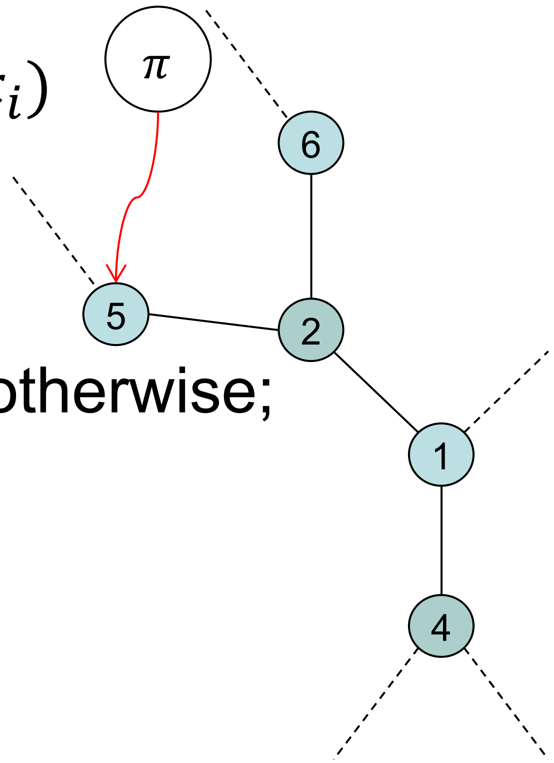
$$\lim_{t\to\infty} x_i(t) = s(t) \ \forall i$$

- ❑ Converging towards a common (synchronous) trajectory
- ❑ Imposing this trajectory (the so called pinning control problem)

Pinning control

$$\dot{x}_i = f(x_i) + c\sum_j a_{ij}\left(h(x_j) - h(x_i)\right) + \delta_i \kappa_s (s - x_i)$$

$$\dot{s} = f(s)$$



Where

- $\delta_i \in \{0,1\}$ and takes the value of 1 if node i is *pinned* and 0 otherwise;
- κ_i is a control gain.

- ❑ Converging towards a common (synchronous) trajectory
- ❑ Imposing this trajectory (the so called pinning control problem)

Network Controllability

$$\dot{x}_i = f(x_i, u_i) + \sum_j h(x_i, x_j)$$

Some strategies have been proposed, none with guarantees to the best of my knwoledge

…however…

$$\dot{x}_i = \sum_j a_{ij} x_j + \sum_l b_{il}\, u_l$$

i. We know how to select b_{il} so to ensure controllability of a network [5,6]
ii. we have studied the relationship between the number of driver nodes and the control effort [7,8]

- Reaching a predefined point of the state space

My contributions

Sometimes it is impossible or unnecessary to control all the network nodes!

<u>Problem Formulation:</u>

Select the nodes where to inject a fixed number of input signals so to maximize

i. the number of controllable nodes (in Kalman's sense) [9]
ii. the number of *pinning controllable* nodes [10].

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Select the nodes where to inject a fixed number of input signals so to maximize

i. the number of controllable nodes (in Kalman's sense) [9]

ii. the number of *pinning controllable* nodes [10].

This formulation is obsolete as maximizing the ratio between controllable and driver nodes leads to network control being energetically prohibitive.

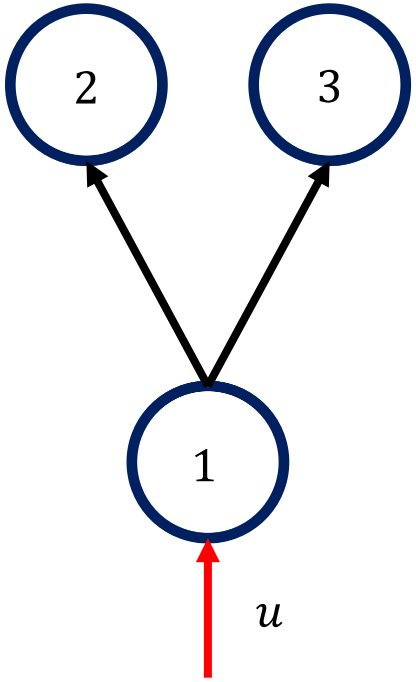
i. What is a set of controllable nodes?

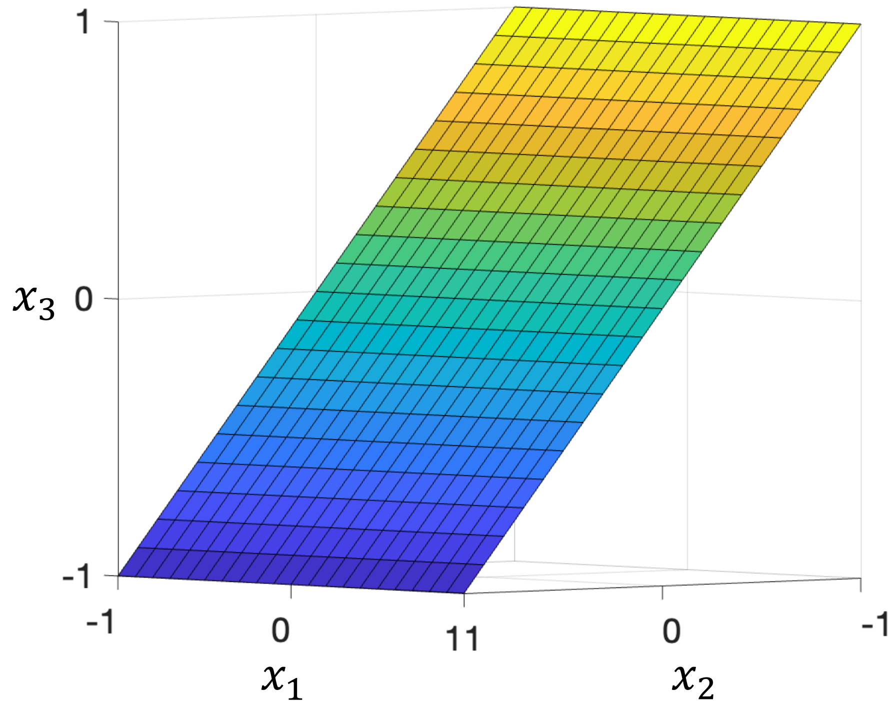
ii. What is a set of observable nodes?

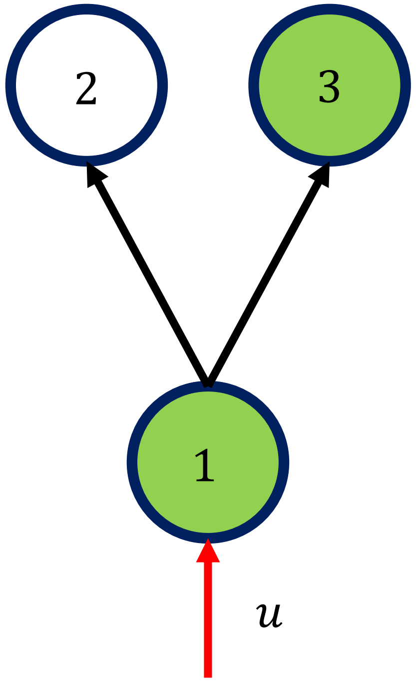
Node Controllability

What is a set of controllable nodes?

The controllable subspace might not be the span of a subset of the columns of the identity matrix.







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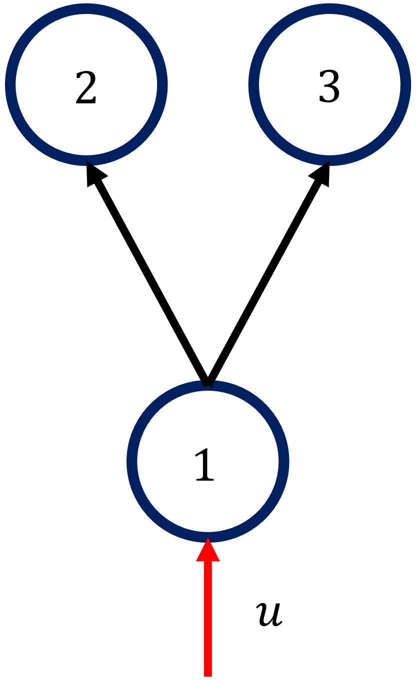
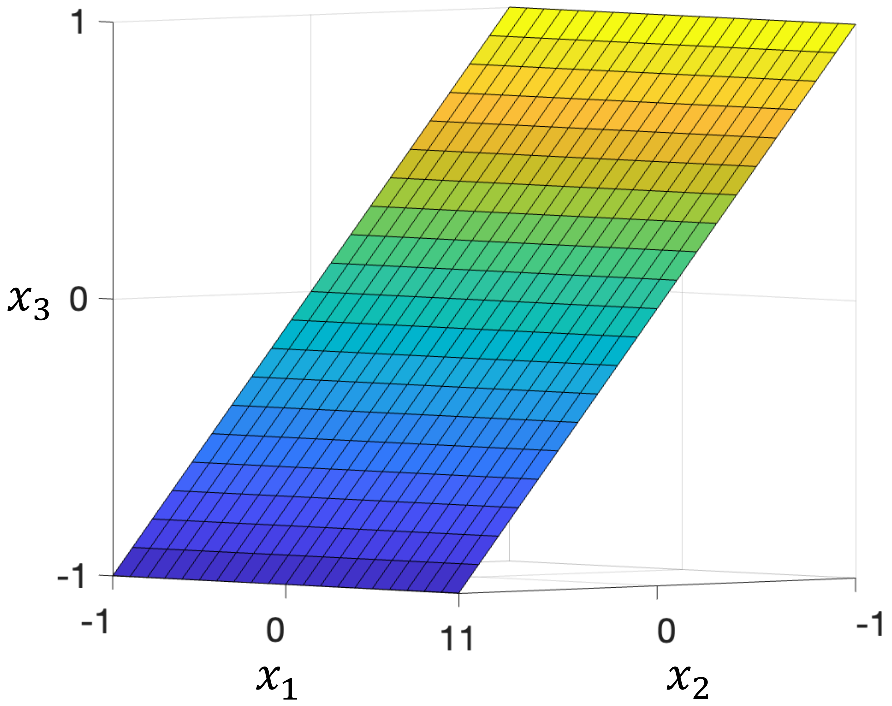
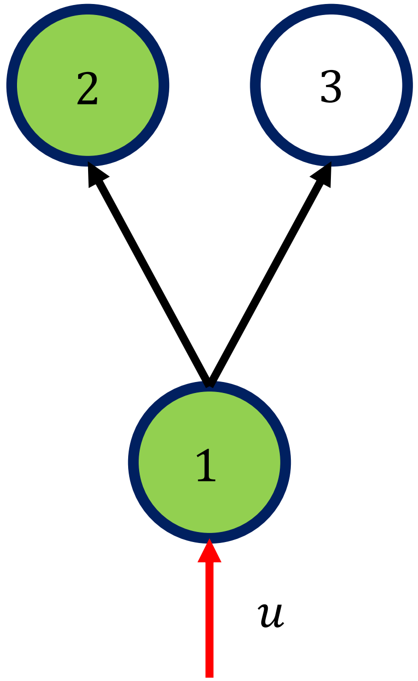


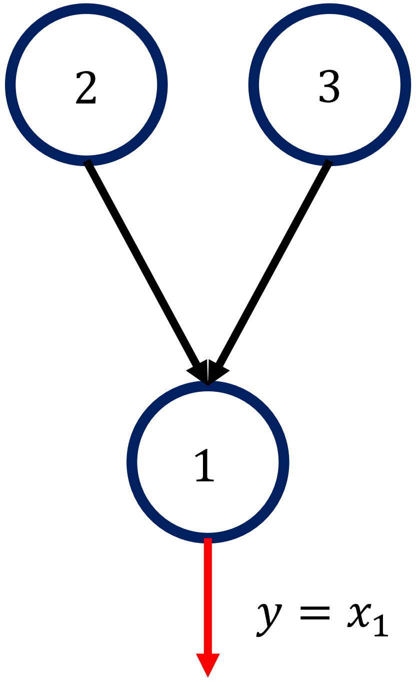


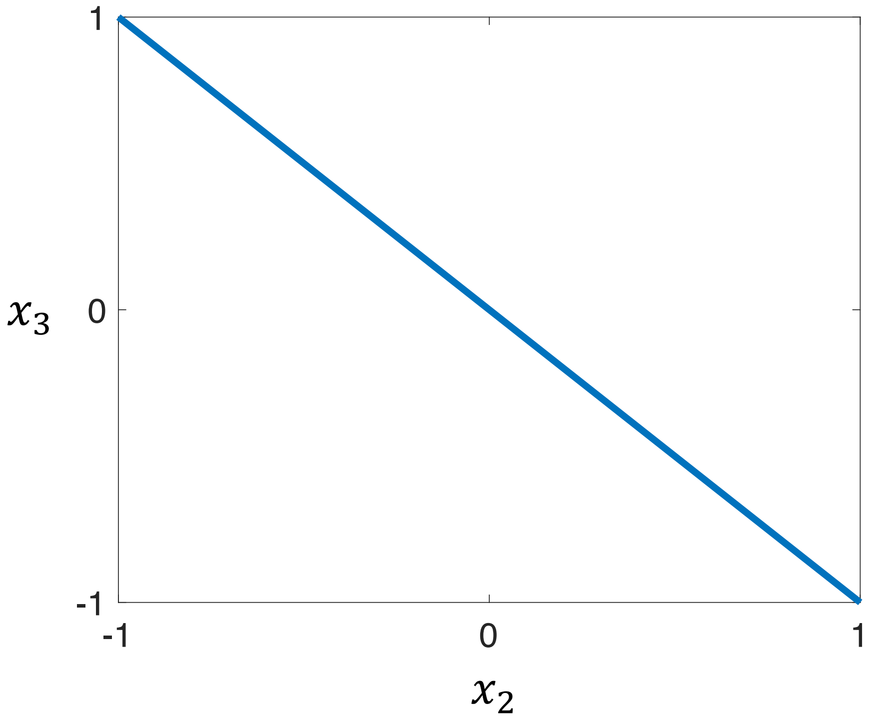


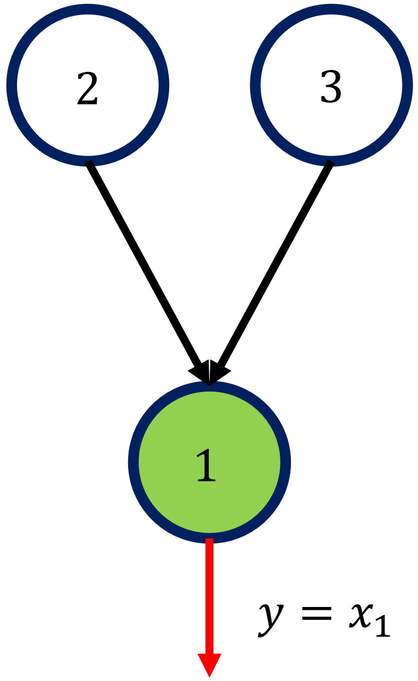
Node Observability

What is a set of observable nodes?

For dynamical systems we define the un-observable subspace.







On node controllability and observability [11]

Sometimes it is impossible or unnecessary to control all the network nodes!

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This formulation is obsolete as maximizing the ratio between controllable and driver nodes leads to network control being energetically prohibitive.

1. The set of controllable nodes might not be unique;
2. Some non controllable nodes might still be perturbed by the control action;
3. The set of observable nodes can be smaller than the dimension of the complement to the observable subspace.

Partial Pinning Control [10]

Sometimes it is impossible or unnecessary to control all the network nodes!

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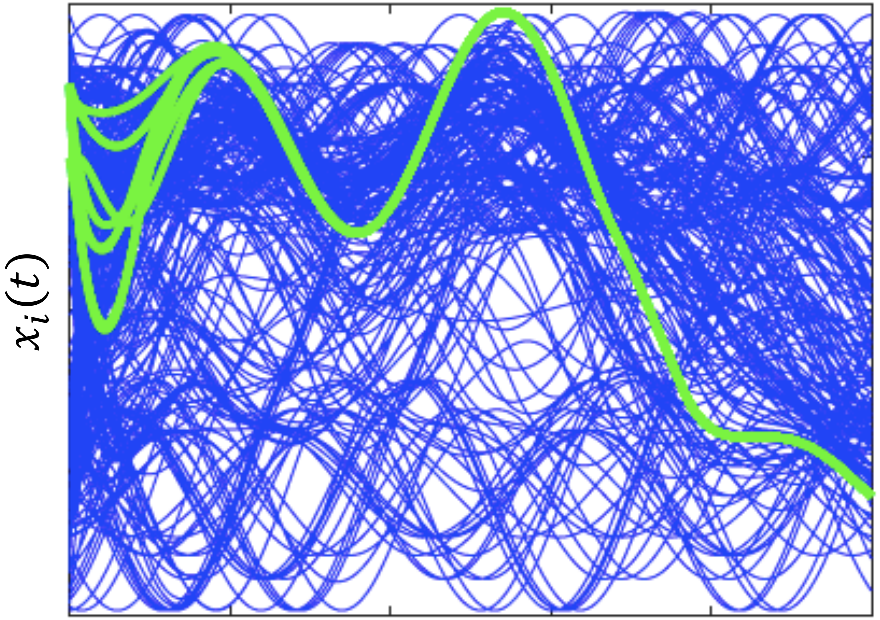

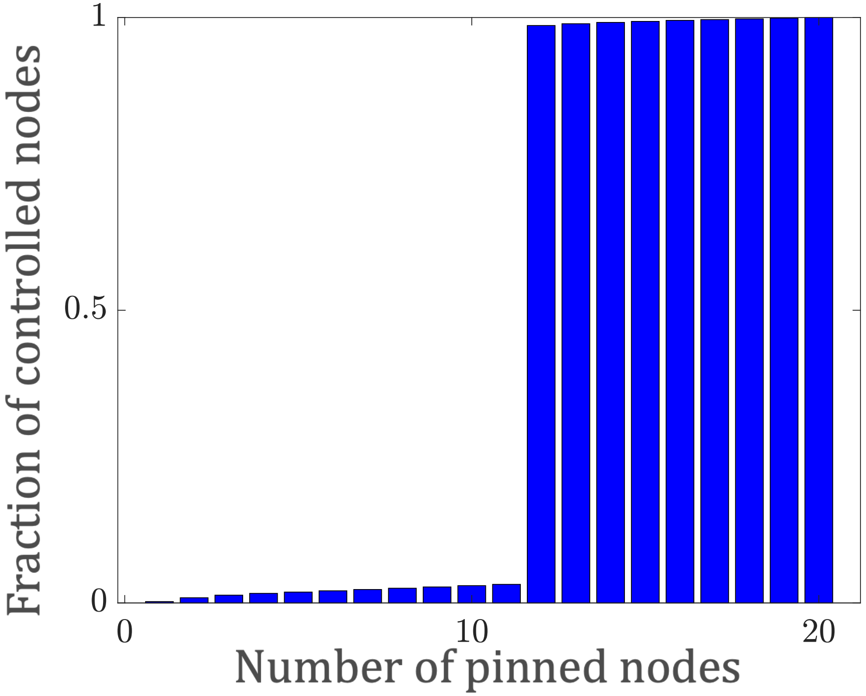

From pinning control to opinion dynamics

Sometimes it is impossible or unnecessary to control all the network nodes!

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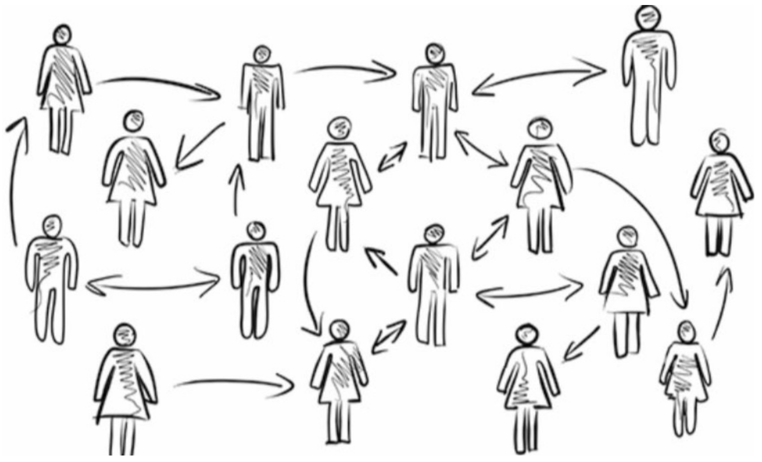
Can we exploit this framework to model the role of influencers in social networks?

Opinion Dynamics Of Networked Agents

Networked Opinion dynamics models describe how N agents shape their opinions x_i by interacting over a social network leading to the emergence of collective behavior.



Most opinion dynamics models all stem from the linear consensus protocol [12]

$$x_i(k+1) = \sum_{j=1}^{N} \alpha_{ij} x_j(k) \;\Rightarrow\; x(k+1) = Ax(k)$$

where $A = \{a_{ij}\}_{i=1}^{N}$ is associated to the graph $\mathcal{G}$.

Both discrete time and continuous time models have been proposed.

Limitations of the classical opinion dynamics models

In our view their main limitations are that

- Heterogeneous opinions emerge only as a result of heterogeneous node dynamics
- Clusters of opinions emerge either from state dependent and initially disconnected graphs [13], or from strong structural assumptions [14]
- When clusters of opinions do emerge, all opinions in the same cluster are the same
- The presence of influencers is not modelled (in this literature).

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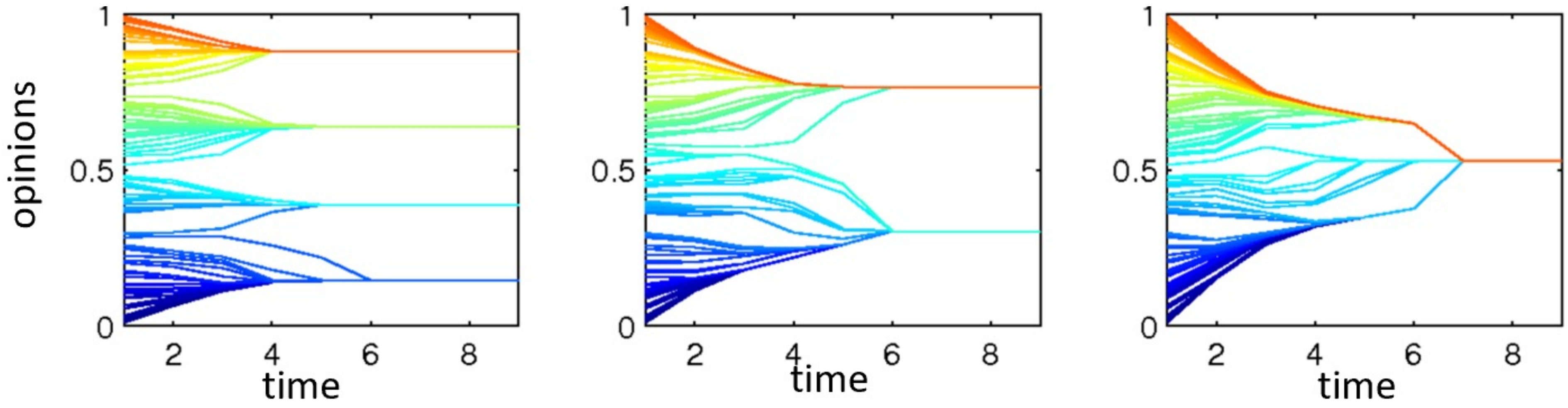

Merging nonlinear OD with pinning control

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These limitations can be overcome by considering existing nonlinear opinion dynamics models superimposed on directed graphs [15,16,17].

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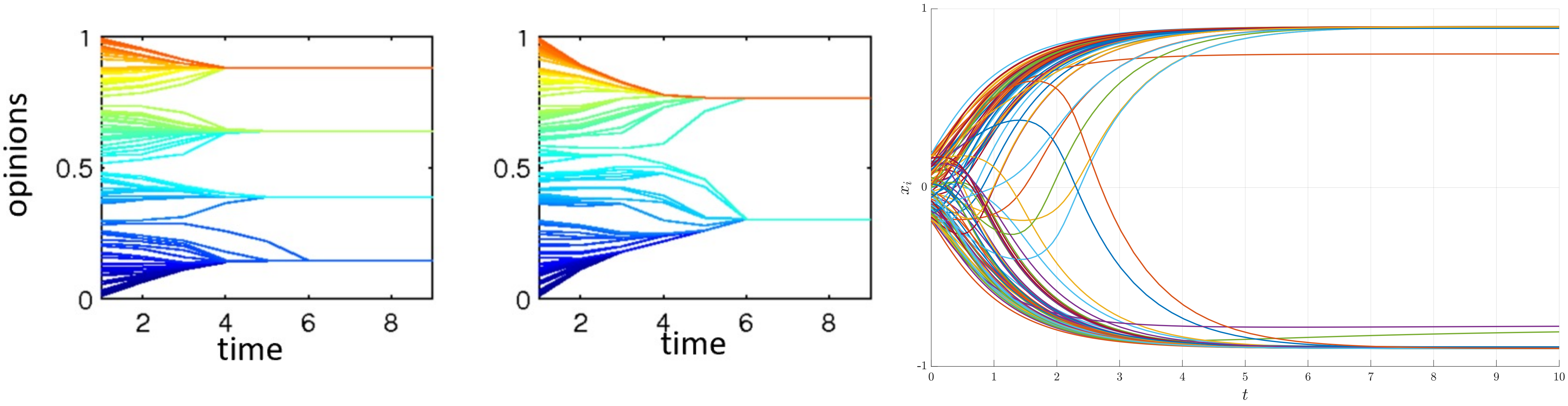

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The role of influencers can be modelled as pinning control actions.

Adapting the pinning control strategy to nonlinear opinion dynamics models required the development of some novel theoretical results.

Bounded Partial Pinning Control [18]

We consider a set of N agents on a directed graph and whose dynamics are

$$\dot{x}_i = f(x_i) + \sigma \sum_j a_{ij}(h(x_j) - h(x_i))$$

Where the functions $f(\cdot)$ and $h(\cdot)$ satisfy the quad-like assumption

$$(x - z)^T\, V\left(f(x) - f(z)\right) \leq (z - x)^T\, W(z - x)$$

$$W - \mu\, VH \; < -\tau\, I_n, \quad VH = H^{\mathrm{T}} V^{\mathrm{T}} \geq \; 0$$

$$(x - z)^{\mathrm{T}}\, V\big(h(x) - h(z)\big) \geq \; (x - z)^{\mathrm{T}}\, H\,(x - z)$$

$$\|h(z) - h(x)\| \leq l||z - x||$$

For some matrices V, W, H and some scalars τ, μ, l

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We consider a set of N agents on a directed graph and whose dynamics are

$$\dot{x}_i = f(x_i) + \sigma \sum_j a_{ij}(h(x_j) - h(x_i))$$

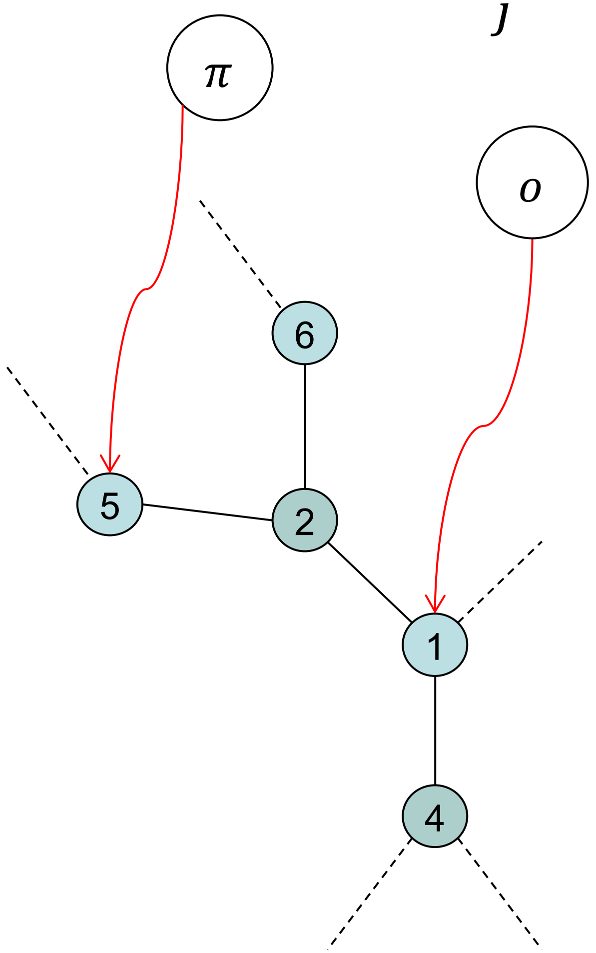
We assume that the network agents are subject to two competing influencers, whose persuading action is modeled as a pinning signal yielding

$$\dot{x}_i = f(x_i) + \sigma \sum_j a_{ij}(h(x_j) - h(x_i)) + c_\pi \, \delta_{\pi,i} \, (h(x_i) - h(x_\pi)) + c_o \, \delta_{o,i} \, (h(x_i) - h(x_o))$$

Where x_π and x_o are the states of the two influencers and their trajectories are bounded.

Bounded Partial Pinning Control [18]

$$\dot{x}_i = f(x_i) + \sigma \sum_j a_{ij}(h(x_j) - h(x_i)) + c_\pi\, \delta_{\pi,i}\, (h(x_i) - h(x_\pi)) + c_o\, \delta_{o,i}\, (h(x_i) - h(x_o))$$



The binary variables δ_π and δ_o determine which nodes are directly influenced by the pinner and which by the opponent.

In this illustrative example we have

$$\delta_{\pi,5} = 1$$
$$\delta_{o,i} = 0\ \forall i \neq 1$$
$$\delta_{o,1} = 1$$

Research question [18]

$$\dot{x}_i = f(x_i) + \sigma \sum_j a_{ij}(h(x_j) - h(x_i)) + c_\pi \, \delta_{\pi,i} \, (h(x_i) - h(x_\pi)) + c_o \, \delta_{o,i} \, (h(x_i) - h(x_o))$$

The state of which nodes will converge sufficiently close to the pinner's trajectory?

$$\lim_{t\to\infty} \sup \|x_i(t) - x_\pi\| \leq \epsilon \, ?$$

Research question [18]

$$\dot{x}_i = f(x_i) + \sigma \sum_j a_{ij}(h(x_j) - h(x_i)) + c_\pi\, \delta_{\pi,i}\, (h(x_i) - h(x_\pi)) + c_o\, \delta_{o,i}\, (h(x_i) - h(x_o))$$

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- The motivating question in the field of opinion dynamics is that we assume each node i faces the question of whether it prefers option π or o.
- We model the preference of each agent through the output function

$$y_i = \arg \min_{\gamma\in\{\pi,o\}} \lim_{t\to\infty} \|x_i(t) - x_\gamma\|$$

Analysis: the DAG condensation

$$\dot{x}_i = f(x_i) + \sigma \sum_j a_{ij}\big(h(x_j) - h(x_i)\big) + c_\pi\, \delta_{\pi,i}\, \big(h(x_i) - h(x_\pi)\big) + c_o\, \delta_{o,i}\, \big(h(x_i) - h(x_o)\big)$$

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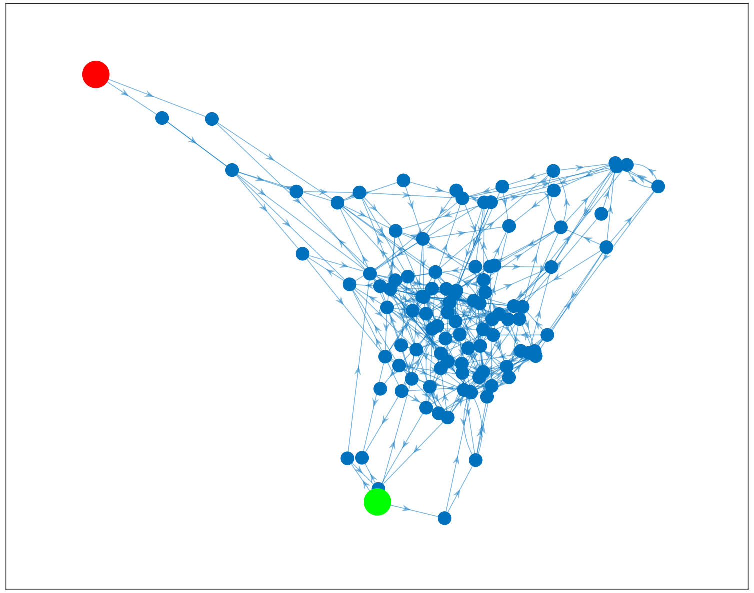
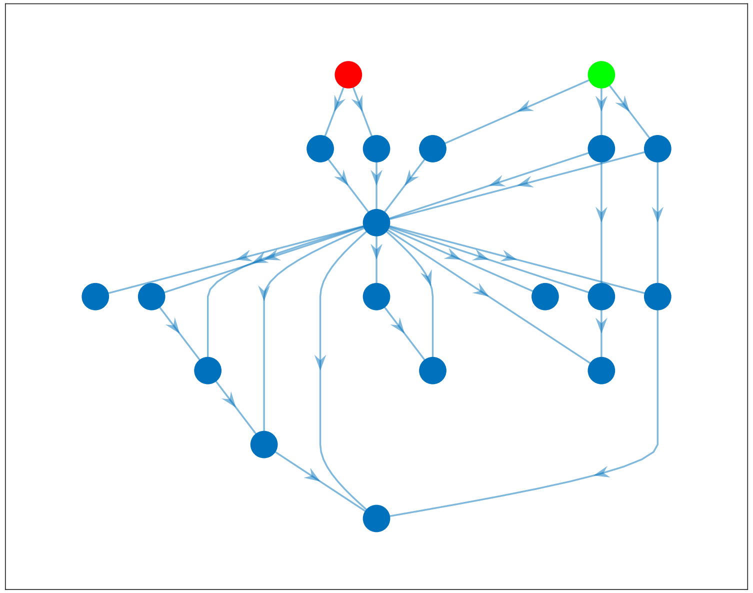


Condensation



Each node of the condensed graph is a Strongly Connected Component of the network graph

- Pinner
- opponent

Analysis: the DAG condensation

$$\dot{x}_i = f(x_i) + \sigma \sum_j a_{ij}\big(h(x_j) - h(x_i)\big) + c_\pi\, \delta_{\pi,i}\, \big(h(x_i) - h(x_\pi)\big) + c_o\, \delta_{o,i}\, \big(h(x_i) - h(x_o)\big)$$

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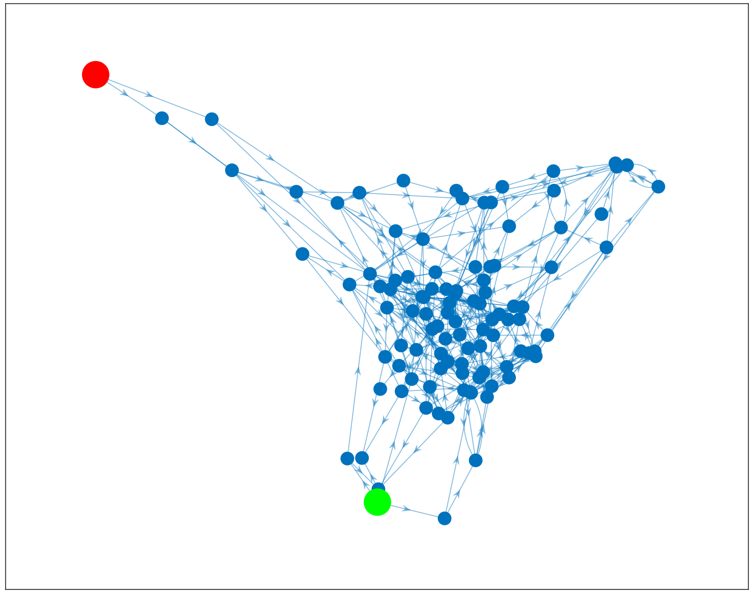

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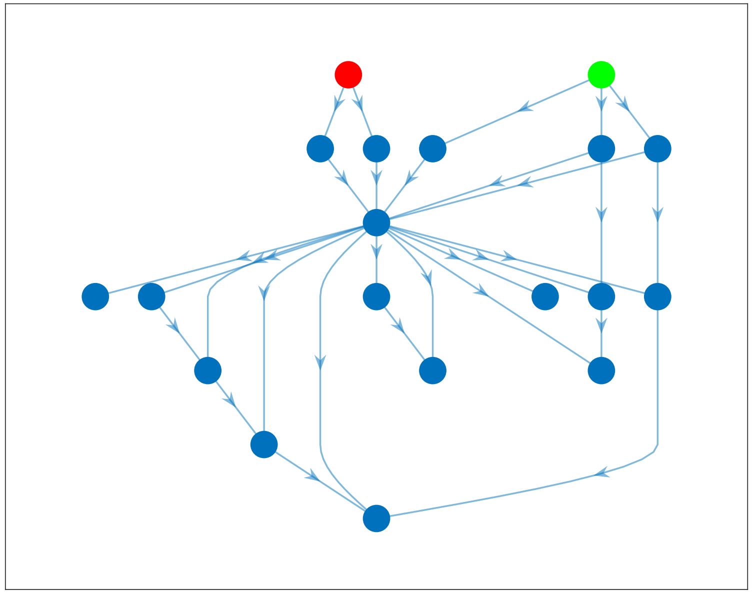

A Strongly Connected Component (SCC) of a graph is a subgraph where there exists a path from any node of the SCC ot any other node of the SCC.

Analysis: bound derivation

$$\dot{x}_i = f(x_i) + \sigma \sum_j a_{ij}(h(x_j) - h(x_i)) + c_\pi\, \delta_{\pi,i}\, (h(x_i) - h(x_\pi)) + c_o\, \delta_{o,i}\, (h(x_i) - h(x_o))$$

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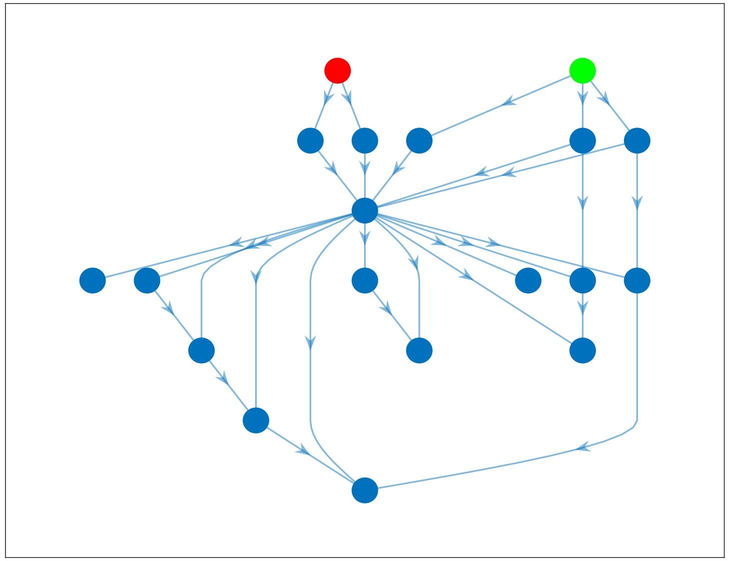

Let G_h be the h-th SCC of the network. Then, for all $i \in V_h$ we show through Lyapunov theory that

$$\lim \sup \|x_i - x_\pi\| \leq \epsilon_h$$

- ϵ_h is made larger by $c_o, |x_\pi - x_o|, V_h^o$
- ϵ_h can be made smaller through c_π, V_h^π

- Pinner
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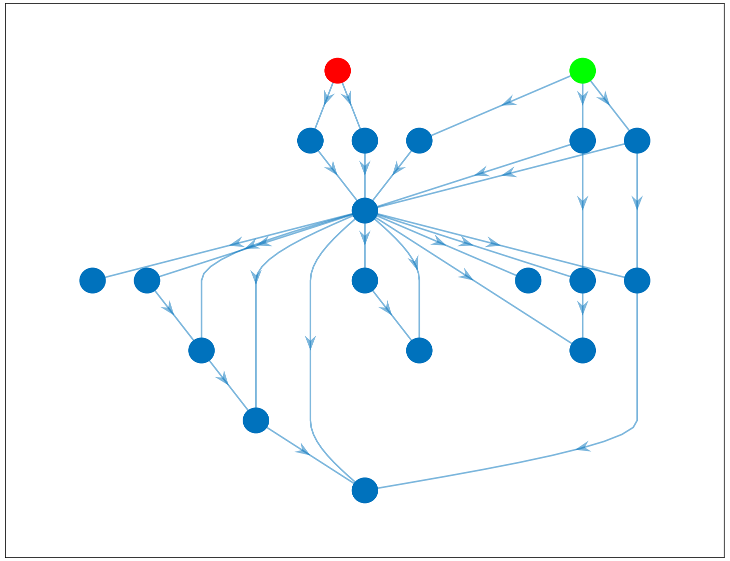

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- ϵ_h can be made smaller through c_π, V_h^π

As we all know, Lyapunov theory always provides conservative conditions. Are these bounds useful?

An opinion dynamics example

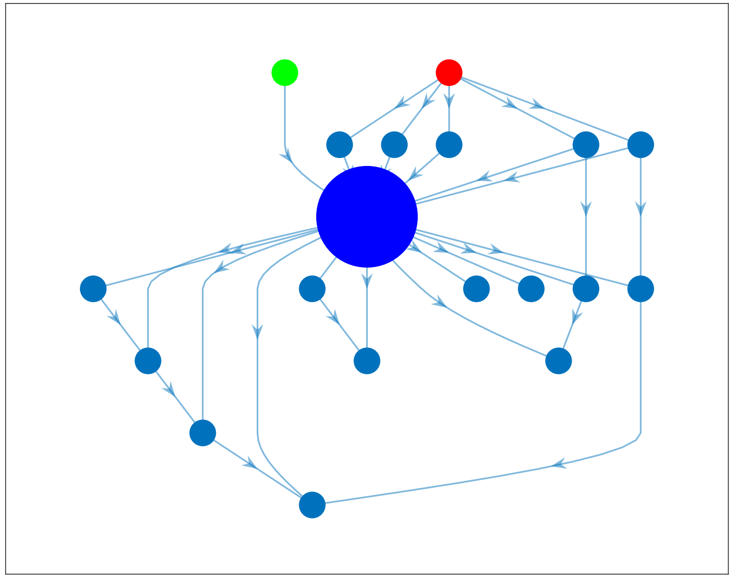
$$\dot{x}_i = f(x_i) + \sigma \sum_j a_{ij}(h(x_j) - h(x_i)) + c_\pi \, \delta_{\pi,i} \, (h(x_i) - h(x_\pi)) + c_o \, \delta_{o,i} \, (h(x_i) - h(x_o))$$

We select

- $f(x_i) = -3x_i + 4\tanh(x_i)$, modeling a bistable system whose to stable equilibria are $(-1,1)$;
- $h(x) = x$;
- A network topology with 14 roots;
- $x_\pi(t) = 1$ and $x_o(t) = -1$ for all t;
- We assume the opponent pins all 14 roots.

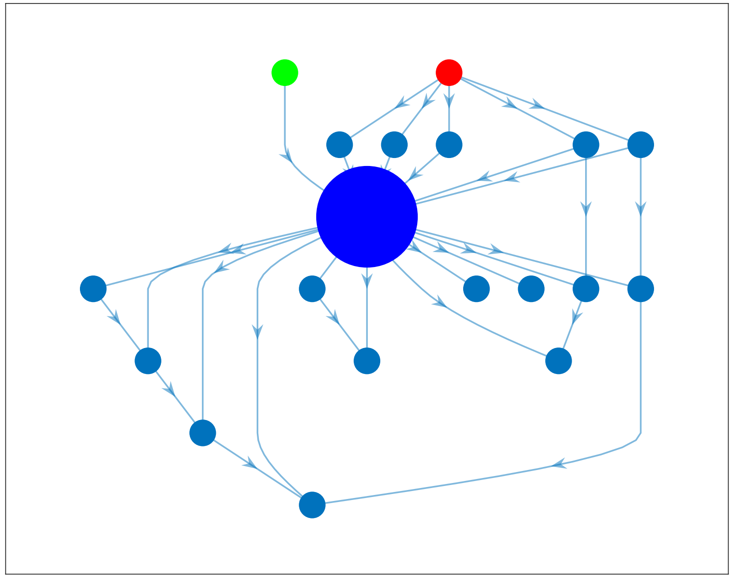
Can a smart selection of the 14 pinned nodes allow the pinner to bring the majority of the network nodes closer to x_π than to x_o?

A simple heuristic



This is a quite challenging scenario as the 14 nodes in level 0 will converge to x_o.
In turn these 14 nodes directly influence 71 nodes in the Giant SCC.

A simple heuristic



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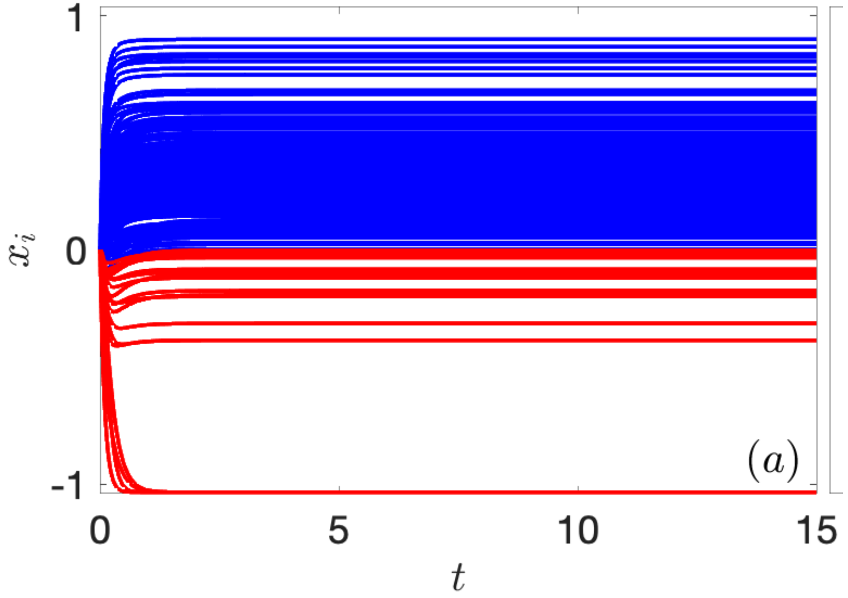

A simple heuristic prescribing to pin the 14 nodes that individually would provide the best bound for the nodes in the Giant SCC yields a surprisingly good result.

Out of 306 agents, 250 ended up voting for the pinner.

Merging nonlinear OD with pinning control

- Heterogeneous opinions emerge only as a result of heterogeneous node dynamics;
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These limitations can be overcome by considering existing nonlinear opinion dynamics models superimposed on directed graphs [15,16,17].

The role of influencers can be modelled as pinning control actions.

$$\dot{x}_i = -dx_i + uS\left(\alpha_i x_i + \sum_j a_{ij} x_j\right)$$

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Our assumptions do not fit the saturated interaction protocols in [16]

$$\dot{x}_i = f(x_i) + \sigma \sum_j a_{ij}(h(x_j) - h(x_i))$$

$$(x-z)^T V \left(f(x) - f(z)\right) \leq (z-x)^T W (z-x)$$

$$W - \mu VH \ < -\tau I_n, \quad VH = H^{\mathrm{T}} V^{\mathrm{T}} \geq \ 0$$

$$(x-z)^{\mathrm{T}} V\left(h(x) - h(z)\right) \geq \ (x-z)^{\mathrm{T}} H \,(x-z)$$

$$\|h(z) - h(x)\| \leq l||z-x||$$

Recently [19] we have extended our results to the dynamics in [16].

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$$\dot{x}_i = -dx_i + uS\left(\alpha_i x_i + \sum_j a_{ij}x_j + \boldsymbol{\delta_i \kappa_s x_s}\right)$$

Centrality of the pinned nodes

Both for controllability and pinning control, the nodes that drive the networks are characterized by small values of most centrality metrics

Considering nonlinear opinion dynamics models and toning down the control goal yields a substantially different result.

	Network	**Pinned Nodes**
Average Indegree	8.03	25.62
Average Outdegree	8.03	26.31
Average betweeness	1022	6303

Conclusions and future work

In the last decade, as a community, we made strides in network control.

I hope I managed to contribute with a novel perspective, that of controlling only a fraction of the network nodes and by linking network control to opinion dynamics.

Plenty of problmes still to be tackled!

1. We still lack synthesis tools to steer networks towards a desired state;
2. Plenty of existing opinion dynamics models, very few tuned on data [20].
3. Can we incorporate the framework of higher order interactions without losing the intuition on the role of the network structure?

References

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Thank you for your attention.