Elliit Focus Group Linköping University 12/9/23

# Turing pattern formation on higher-order networks

Riccardo Muolo
Department of Mathematics & naXys Université de Namur (Belgium)
Department of Systems and control engineering, Tokyo Institute of Technology (Japan)

funded by

# My background

BSc Physics and MSc Applied Mathematics
Firenze (Italy)

PhD Systems Biology (1 year - quit)
Amsterdam (The Netherlands)

Teaching assistant Mathematics

PhD Applied Mathematics

Since May 2023 PostDoc
Namur (Belgium)

Visiting Researcher

From (May) Oct 2023 → PostDoc

# Université de Namur - naXys

Prof. Teo Carletti

group web page

Dynamics on networks and beyond group

pattern-formation, random walks, synchronization, spectral machine learning and more


Namur Institute for Complex Systems

# Tokyo Institute of Technology

Prof. Hiroya Nakao

group web page

Department of Systems
and Control Engineering

phase reduction of weakly coupled oscillators,
phase reconstruction,
quantum nonlinear dynamics,
pattern formation and synchronization,
control of stochastic systems
and more

東京工業大学
Tokyo Institute of Technology

naxys
Namur Institute for Complex Systems

# Outline

Dynamical systems on networks

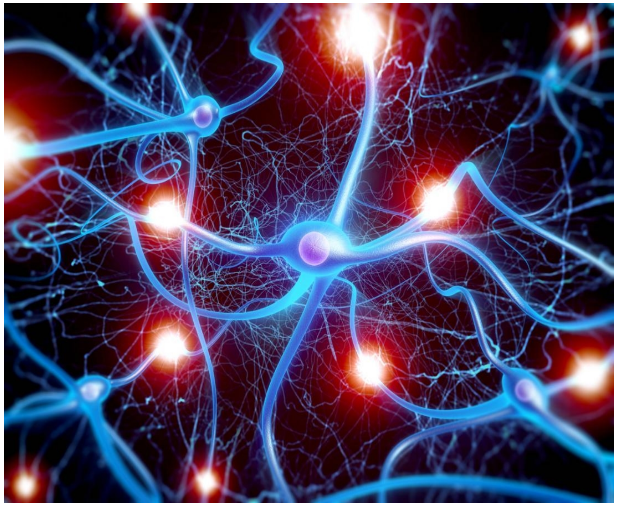
Turing theory of pattern formation

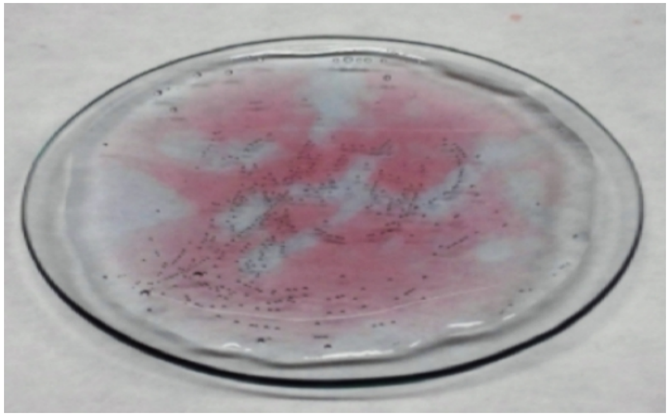
Turing theory on networks

Higher-order structures

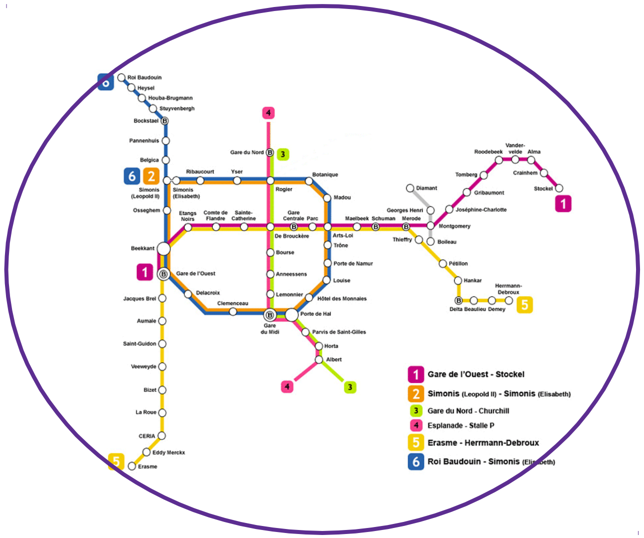
Topological signals

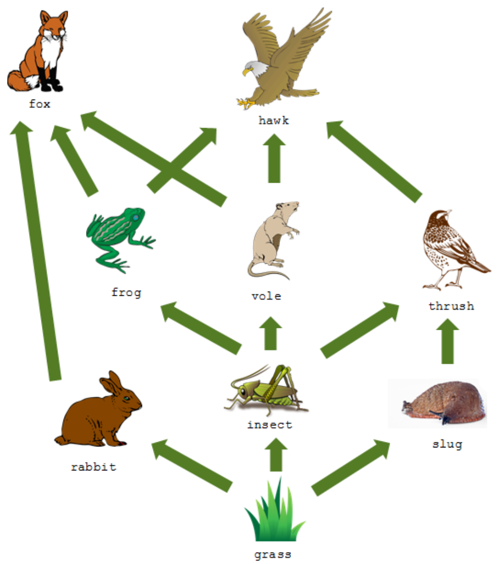
# Patterns in nature



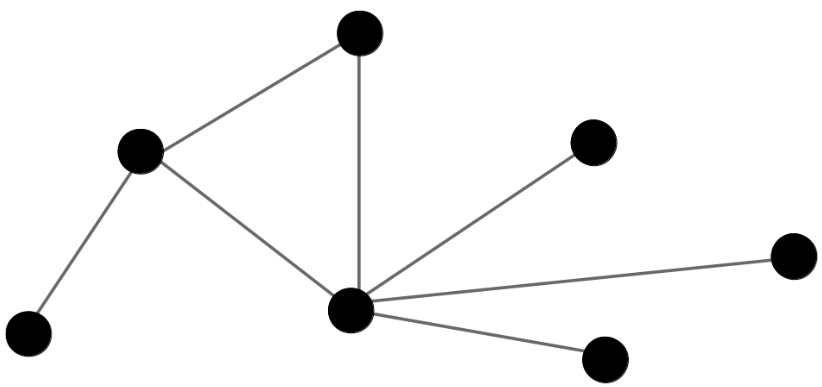


# Networks



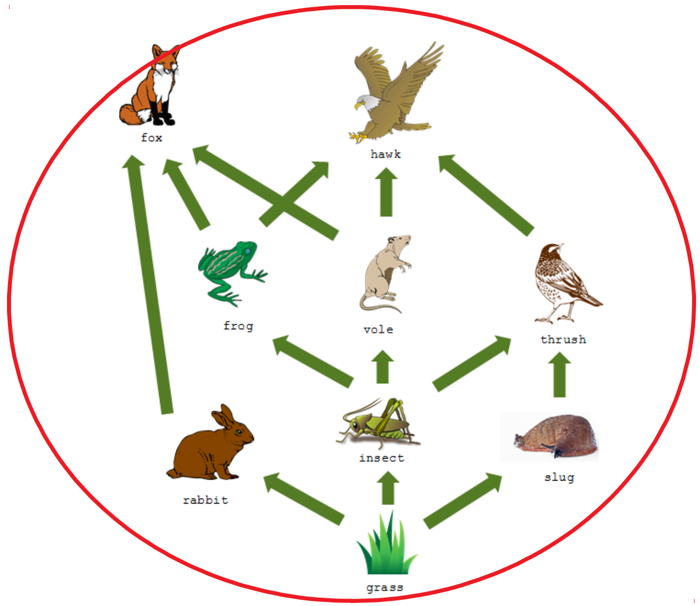

symmetric (undirected)



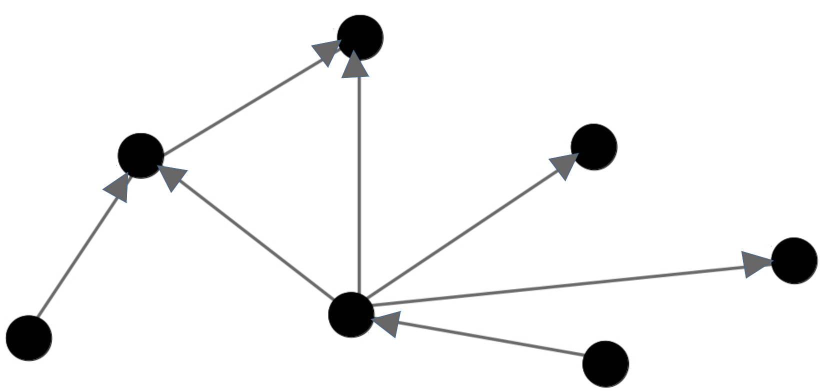
$$A_{ij} = \begin{cases} w_{ij} \in \mathbb{R}^+ \\ 0 \end{cases}$$

if there is a link
between nodes
i and j

# Networks


asymmetric (directed)



$$A_{ij} = \begin{cases} w_{ij} \in \mathbb{R}^+ \\ 0 \end{cases}$$

if there is a directed link
from node j
to node i

# Coupled nonlinear systems

$$\dot{\vec{x}}_i = \vec{f}(\vec{x}_i)$$

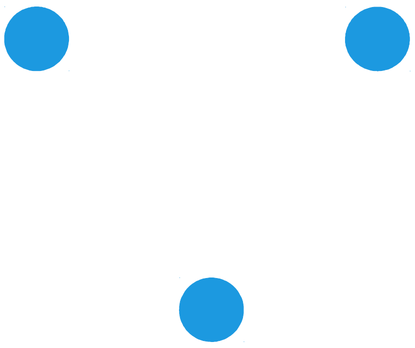
naxys Namur Institute for Complex Systems


riccardo.muolo@unamur.be

# Coupled nonlinear systems

$$\dot{\vec{x}}_i = \vec{f}(\vec{x}_i)$$

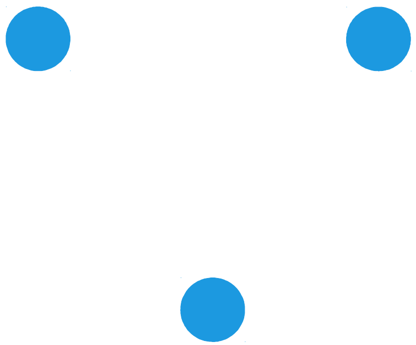
dynamics of $x_i$



# Coupled nonlinear systems

$$\dot{\vec{x}}_i = \vec{f}(\vec{x}_i) + \sigma_1 \sum_j A_{ij}^{(1)} \vec{g}^{(1)}(\vec{x}_i, \vec{x}_j)$$

dynamics of $x_i$

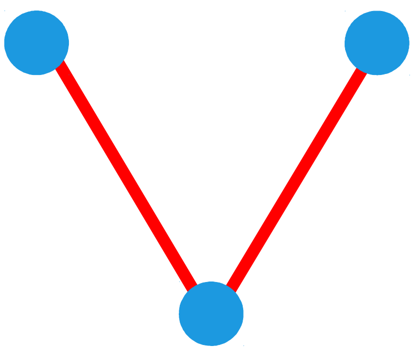


# Coupled nonlinear systems

$$\dot{\vec{x}}_i = \vec{f}(\vec{x}_i) + \sigma_1 \sum_j A_{ij}^{(1)} \vec{g}^{(1)}(\vec{x}_i, \vec{x}_j)$$

dynamics of $x_i$

pairwise coupling



naxys Namur Institute for Complex Systems

# Master Stability Function (MSF)

VOLUME 80, NUMBER 10 PHYSICAL REVIEW LETTERS 9 MARCH 1998

**Master Stability Functions for Synchronized Coupled Systems**

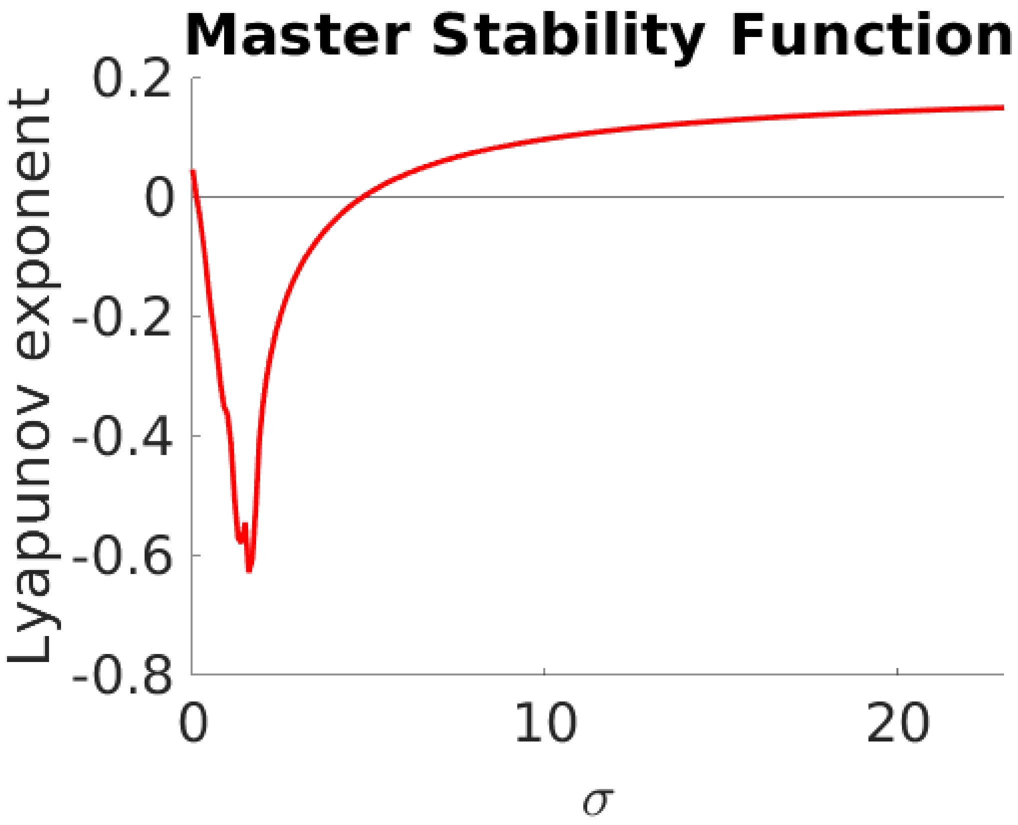
Louis M. Pecora and Thomas L. Carroll
*Code 6343, Naval Research Laboratory, Washington, D.C. 20375*
(Received 7 July 1997)

$$\dot{\xi} = [I_N \otimes J_f + \kappa L \otimes J_h]\xi$$

$$\dot{\xi}_\alpha = [J_f + \kappa \Lambda^{(\alpha)} J_h]\xi_\alpha$$

Rössler

$$\begin{cases} \dot{x}_i = -y_i - z_i + \kappa \sum_j L_{ij} x_j \\ \dot{y}_i = x_i + a y_i \\ \dot{z}_i = b + z_i(x_i - c) \end{cases}$$



# Master Stability Function (MSF)

VOLUME 80, NUMBER 10 PHYSICAL REVIEW LETTERS 9 MARCH 1998

**Master Stability Functions for Synchronized Coupled Systems**

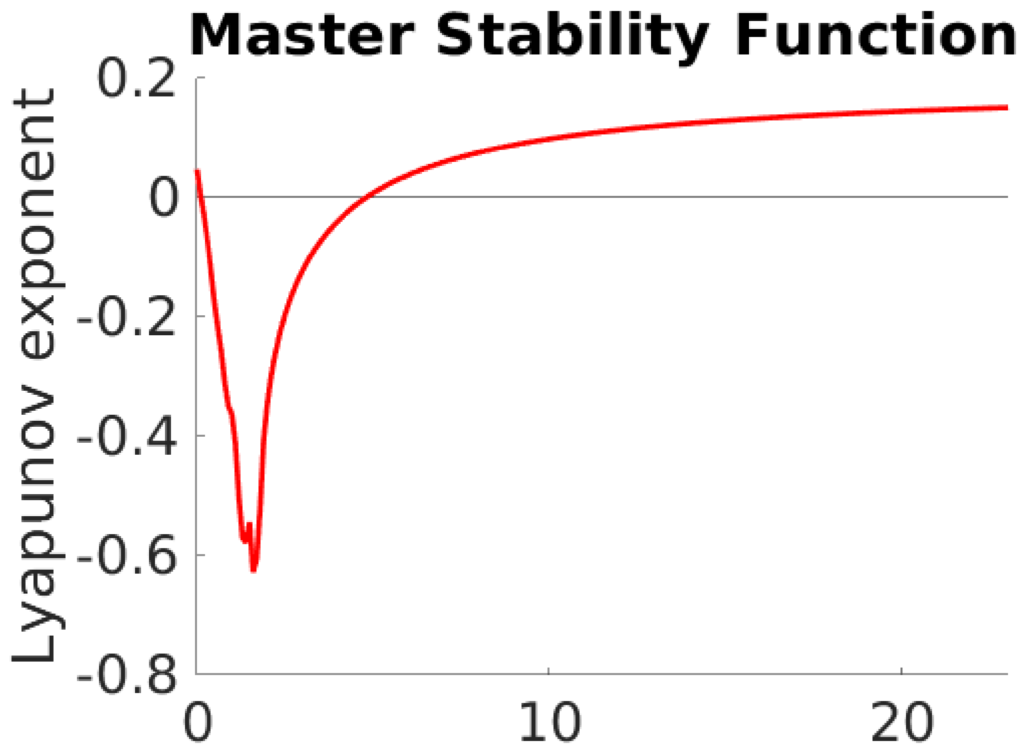
Louis M. Pecora and Thomas L. Carroll
*Code 6343, Naval Research Laboratory, Washington, D.C. 20375*
(Received 7 July 1997)

$$\dot{\xi} = [I_N \otimes J_f + \kappa L \otimes J_h]\xi$$

$$\dot{\xi}_\alpha = [J_f + \kappa \Lambda^{(\alpha)} J_h]\xi_\alpha$$

Rössler

$$\begin{cases} \dot{x}_i = -y_i - z_i + \kappa \sum_j L_{ij} x_j \\ \dot{y}_i = x_i + a y_i \\ \dot{z}_i = b + z_i(x_i - c) \end{cases}$$




$-\Re\Lambda^{(\alpha)}$ $\sigma$

# Master Stability Function (MSF)

VOLUME 80, NUMBER 10 PHYSICAL REVIEW LETTERS 9 MARCH 1998

**Master Stability Functions for Synchronized Coupled Systems**

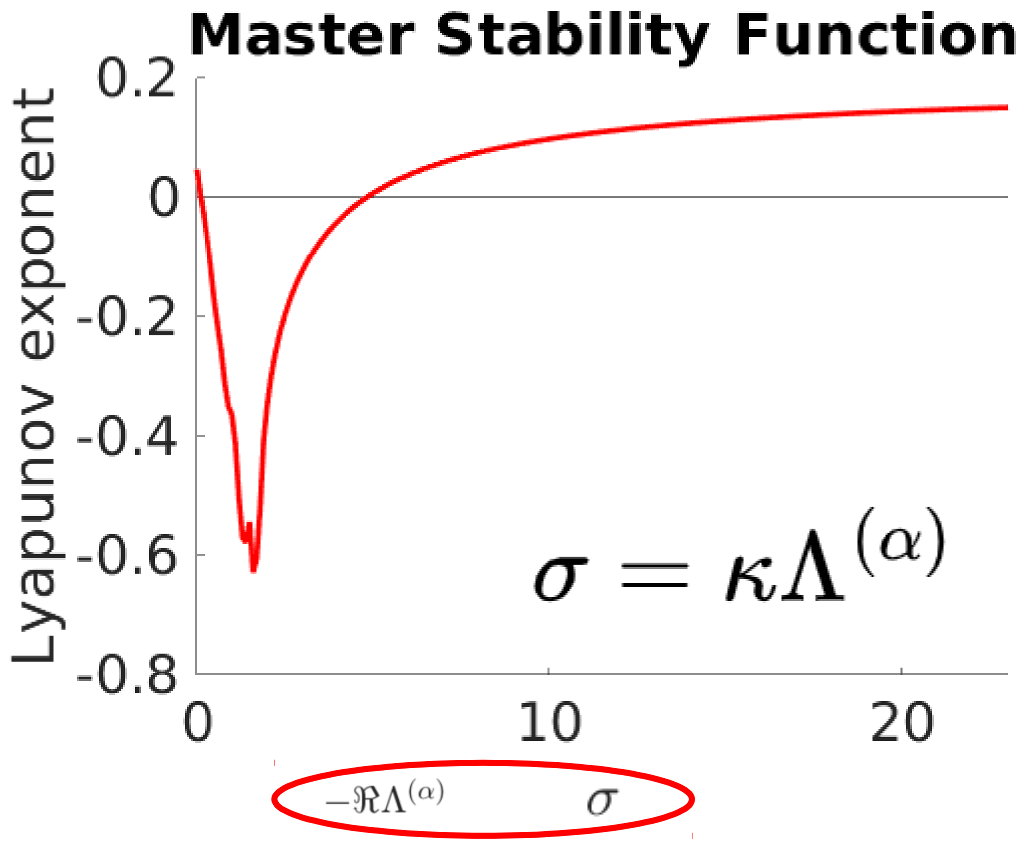
Louis M. Pecora and Thomas L. Carroll
*Code 6343, Naval Research Laboratory, Washington, D.C. 20375*
(Received 7 July 1997)

$$\dot{\xi} = [I_N \otimes J_f + \kappa L \otimes J_h]\xi$$

$$\dot{\xi}_\alpha = [J_f + \kappa \Lambda^{(\alpha)} J_h]\xi_\alpha$$

Rössler

$$\begin{cases} \dot{x}_i = -y_i - z_i + \kappa \sum_j L_{ij} x_j \\ \dot{y}_i = x_i + a y_i \\ \dot{z}_i = b + z_i(x_i - c) \end{cases}$$



# Master Stability Function (MSF)

VOLUME 80, NUMBER 10 PHYSICAL REVIEW LETTERS 9 MARCH 1998

**Master Stability Functions for Synchronized Coupled Systems**

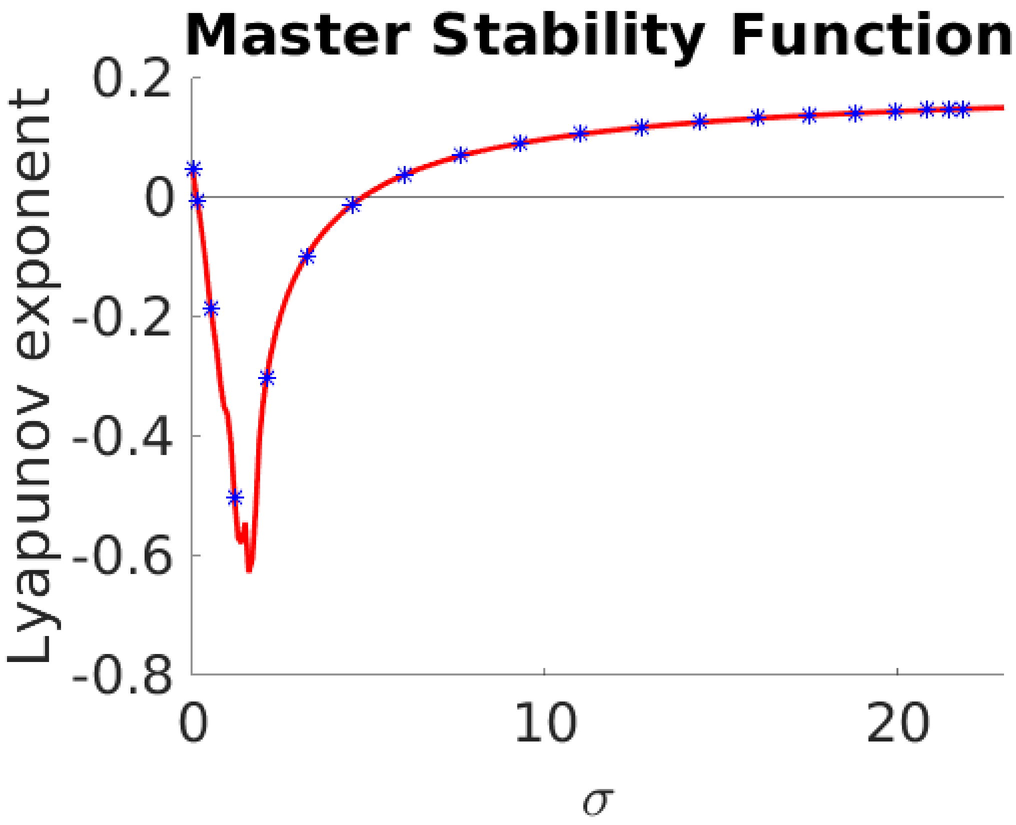
Louis M. Pecora and Thomas L. Carroll
*Code 6343, Naval Research Laboratory, Washington, D.C. 20375*
(Received 7 July 1997)

$$\dot{\xi} = [I_N \otimes J_f + \kappa L \otimes J_h]\xi$$

$$\dot{\xi}_\alpha = [J_f + \kappa \Lambda^{(\alpha)} J_h]\xi_\alpha$$

Rössler

$$\begin{cases} \dot{x}_i = -y_i - z_i + \kappa \sum_j L_{ij} x_j \\ \dot{y}_i = x_i + a y_i \\ \dot{z}_i = b + z_i(x_i - c) \end{cases}$$



# Master Stability Function (MSF)

VOLUME 80, NUMBER 10 PHYSICAL REVIEW LETTERS 9 MARCH 1998

**Master Stability Functions for Synchronized Coupled Systems**

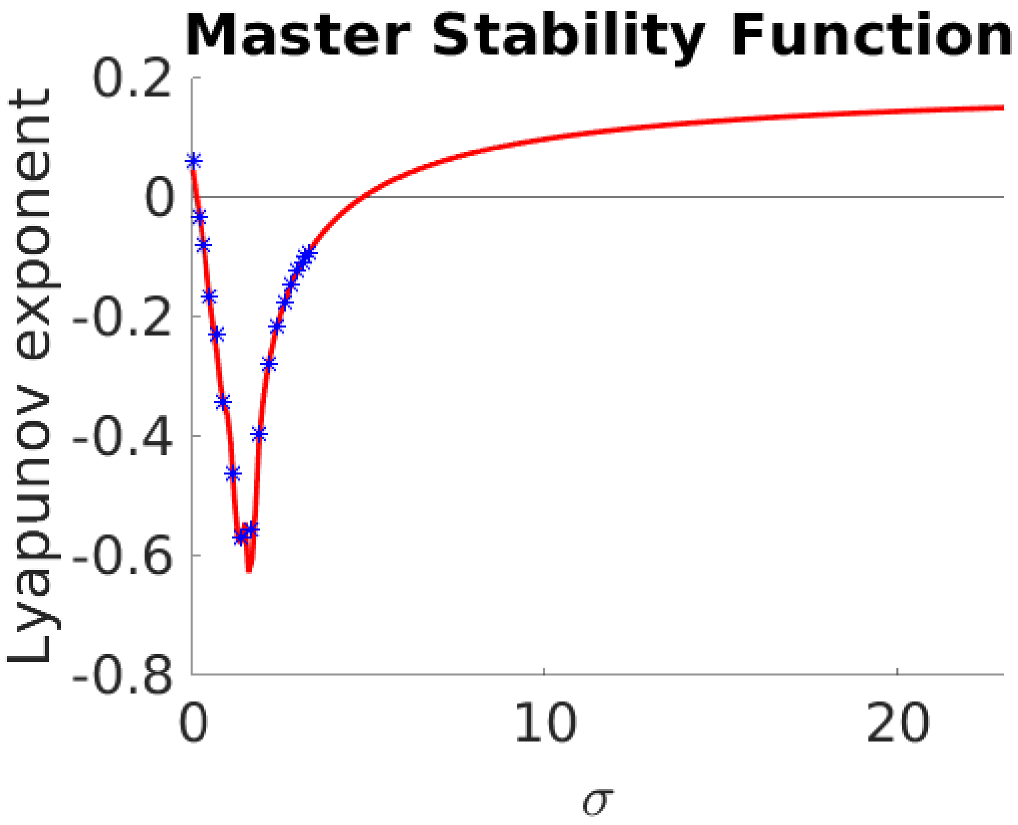
Louis M. Pecora and Thomas L. Carroll
*Code 6343, Naval Research Laboratory, Washington, D.C. 20375*
(Received 7 July 1997)

$$\dot{\xi} = [I_N \otimes J_f + \kappa L \otimes J_h]\xi$$

$$\dot{\xi}_\alpha = [J_f + \kappa \Lambda^{(\alpha)} J_h]\xi_\alpha$$

Rössler

$$\begin{cases} \dot{x}_i = -y_i - z_i + \kappa \sum_j L_{ij} x_j \\ \dot{y}_i = x_i + a y_i \\ \dot{z}_i = b + z_i(x_i - c) \end{cases}$$

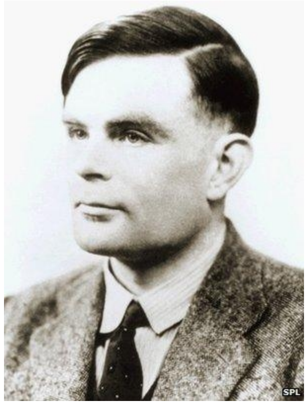


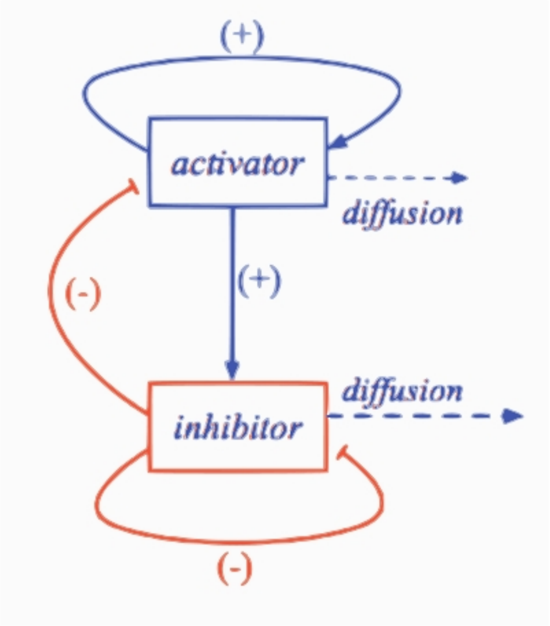
# Intermezzo: Turing theory

**The Chemical Basis of Morphogenesis**

A. M. Turing

*Philosophical Transactions of the Royal Society of London. Series B, Biological Sciences*, Vol. 237, No. 641. (Aug. 14, 1952), pp. 37-72.

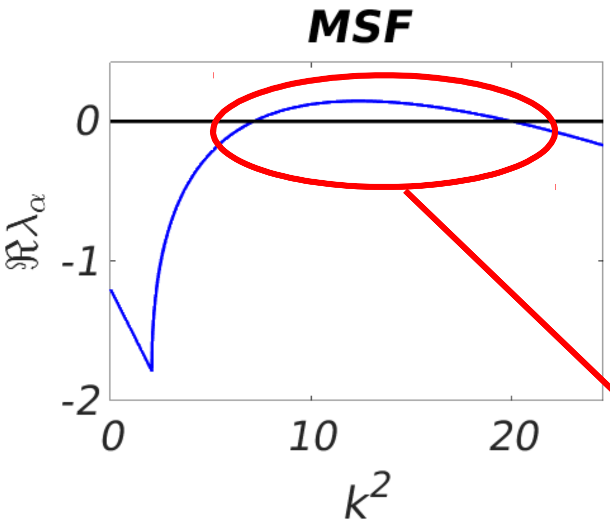



$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = f(u,v) + D_u \nabla^2 u(x,t) \\ \\ \dfrac{\partial v}{\partial t}(x,t) = g(u,v) + D_v \nabla^2 v(x,t) \end{cases}$$

+ boundary conditions
+ domain of the Laplacian
+ ….

# Turing theory in a nutshell




unstable modes

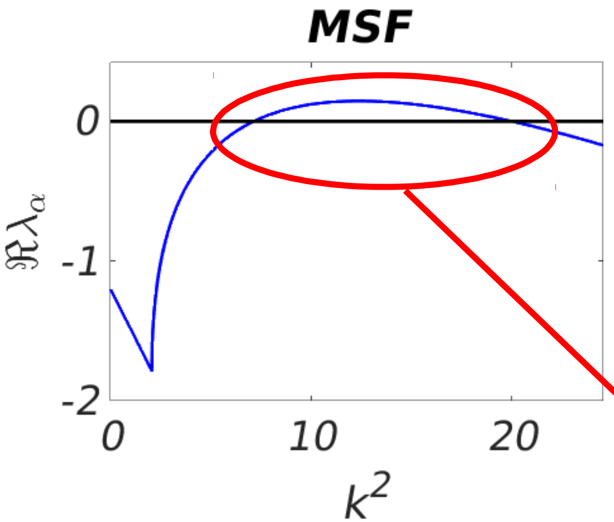
reaction-diffusion system of two species (activator *u* and inhibitor *v*)

homogeneous stable state (fixed point)

inhomogeneous perturbations

exponential instability (diffusion-driven)

patterns $D_v > D_u$




unstable modes

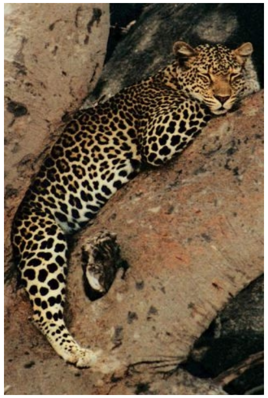
reaction-diffusion system of two species (activator *u* and inhibitor *v*)

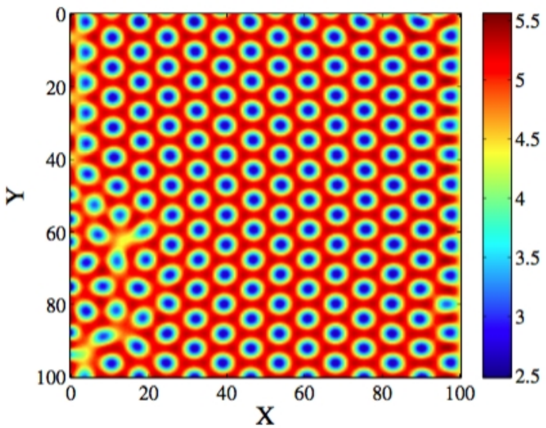
homogeneous stable state (fixed point)

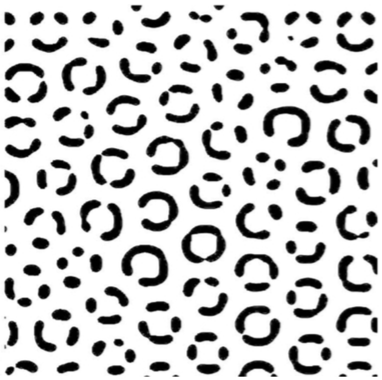
inhomogeneous perturbations

exponential instability (diffusion-driven)

patterns $D_v > D_u$





# Extension on networks

ARTICLES
PUBLISHED ONLINE: 25 APRIL 2010 | DOI: 10.1038/NPHYS1651
nature physics

**Turing patterns in network-organized activator–inhibitor systems**

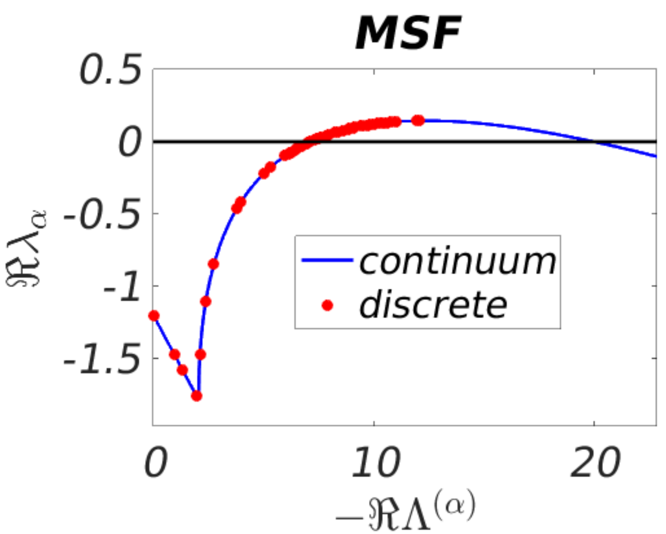
Hiroya Nakao[1,2]* and Alexander S. Mikhailov[3]*

*J. theor. Biol.* (1971) **32,** 507–537

**Instability and Dynamic Pattern in Cellular Networks**

H. G. Othmer† and L. E. Scriven‡

*Department of Chemical Engineering and Materials Science Institute of Technology, University of Minnesota, Minneapolis, Minnesota 55455, U.S.A.*




$$L_{ij} = A_{ij} - k_i\delta_{ij}$$

$$\begin{cases} \dfrac{du_i}{dt} = f(u_i, v_i) + D_u \displaystyle\sum_{j=1}^{nodes} L_{ij}u_j \\ \dfrac{dv_i}{dt} = g(u_i, v_i) + D_v \displaystyle\sum_{j=1}^{nodes} L_{ij}v_j \end{cases}$$

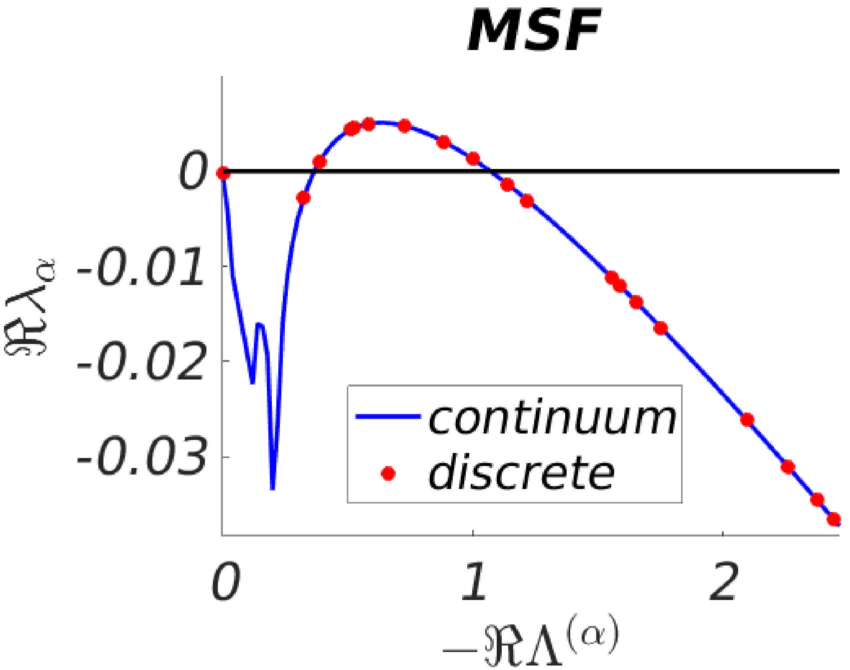
# Turing-like instability

PHYSICAL REVIEW E **92**, 022818 (2015)

**Turing-like instabilities from a limit cycle**

Joseph D. Challenger,[1,2] Raffaella Burioni,[3] and Duccio Fanelli[2]

the homogeneous stable state is a **limit cycle**



# Coupled nonlinear systems (!)

$$\dot{\vec{x}}_i = \vec{f}(\vec{x}_i) + \sigma_1 \sum_j A_{ij}^{(1)} \vec{g}^{(1)}(\vec{x}_i, \vec{x}_j)$$

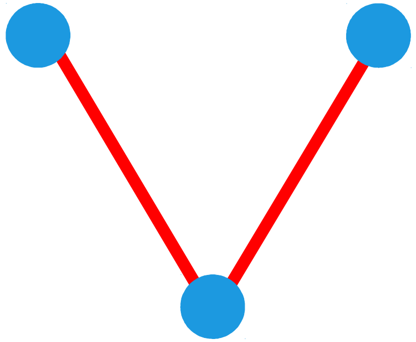
dynamics of $x_i$

chaotic, oscillating,

Fixed point, etc...

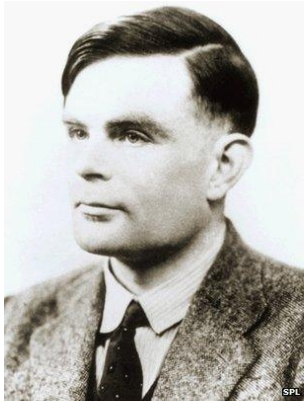
pairwise coupling

in general diffusive like



# Coupled nonlinear systems (!)

$$\dot{\vec{x}}_i = \vec{f}(\vec{x}_i) + \sigma_1 \sum_j A_{ij}^{(1)} \vec{g}^{(1)}(\vec{x}_i, \vec{x}_j)$$



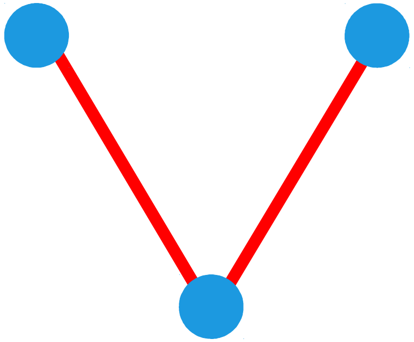

dynamics of $x_i$

chaotic, oscillating,

Fixed point, etc...

pairwise coupling

in general diffusive like



# Extensions of Turing theory

PHYSICAL REVIEW E **92**, 022818 (2015)

## Turing-like instabilities from a limit cycle

Joseph D. Challenger,[1,2] Raffaella Burioni,[3] and Duccio Fanelli[2]

Journal of Theoretical Biology 480 (2019) 81–91

Contents lists available at ScienceDirect

Journal of Theoretical Biology

journal homepage: www.elsevier.com/locate/jtb

Patterns of non-normality in networked systems

Riccardo Muolo[a], Malbor Asllani[b,c,*], Duccio Fanelli[d], Philip K. Maini[b], Timoteo Carletti[e]

nature COMMUNICATIONS

ARTICLE

Received 5 Feb 2014 | Accepted 26 Jun 2014 | Published 31 Jul 2014 DOI: 10.1038/ncomms5517

The theory of pattern formation on directed networks

Malbor Asllani[1,2], Joseph D. Challenger[2], Francesco Saverio Pavone[2,3,4], Leonardo Sacconi[3,4] & Duccio Fanelli[2]

Journal of Physics: Complexity

OPEN ACCESS

PAPER

RECEIVED 27 August 2021

REVISED 29 September 2021

ACCEPTED FOR PUBLICATION 5 October 2021

Finite propagation enhances Turing patterns in reaction–diffusion networked systems

Timoteo Carletti* and Riccardo Muolo

Department of Mathematics & naXys, Namur Institute for Complex Systems, University of Namur, rue Grafé 2, 5000 Namur, Belgium

* Author to whom any correspondence should be addressed.

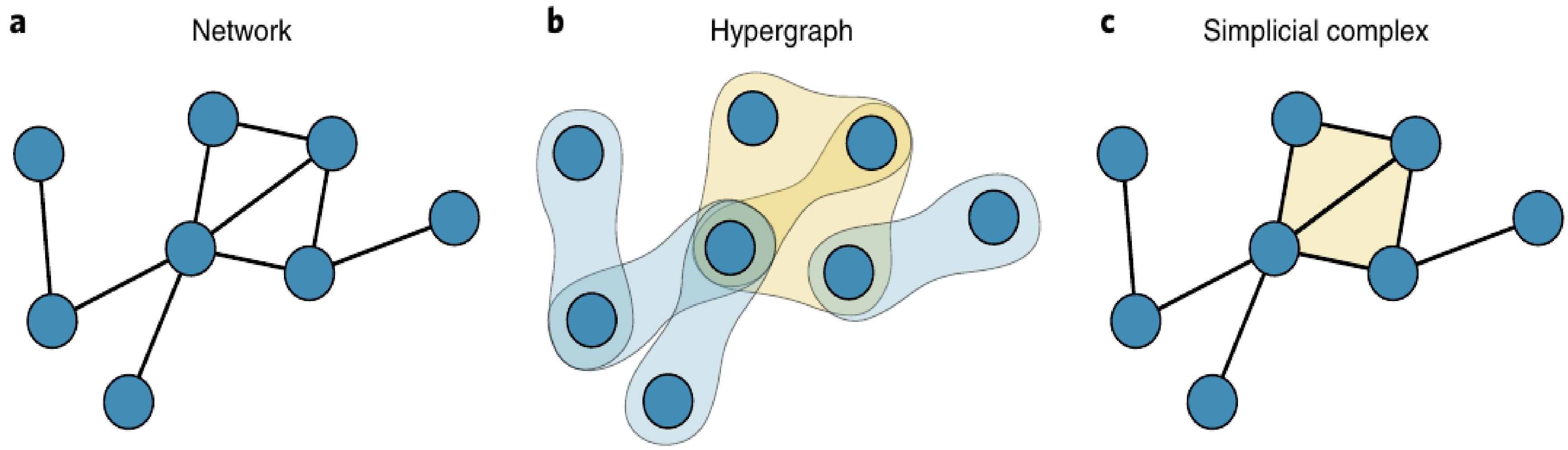
E-mail: timoteo.carletti@unamur.be

Chaos, Solitons and Fractals 134 (2020) 109707

Generalized patterns from local and non local reactions

Giulia Cencetti[a], Federico Battiston[b], Timoteo Carletti[c], Duccio Fanelli[d,*]

# Higher-order Structures

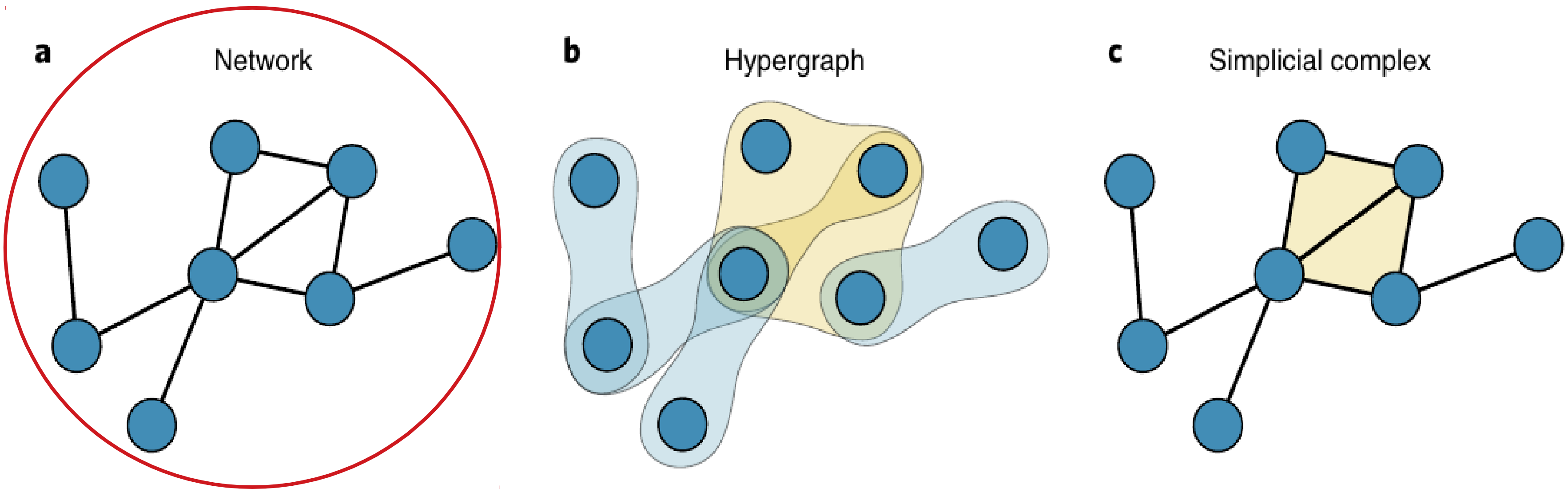



Battiston et al., Nat. Phys., 2021

# Higher-order Structures

adjacency matrix

adjacency tensors

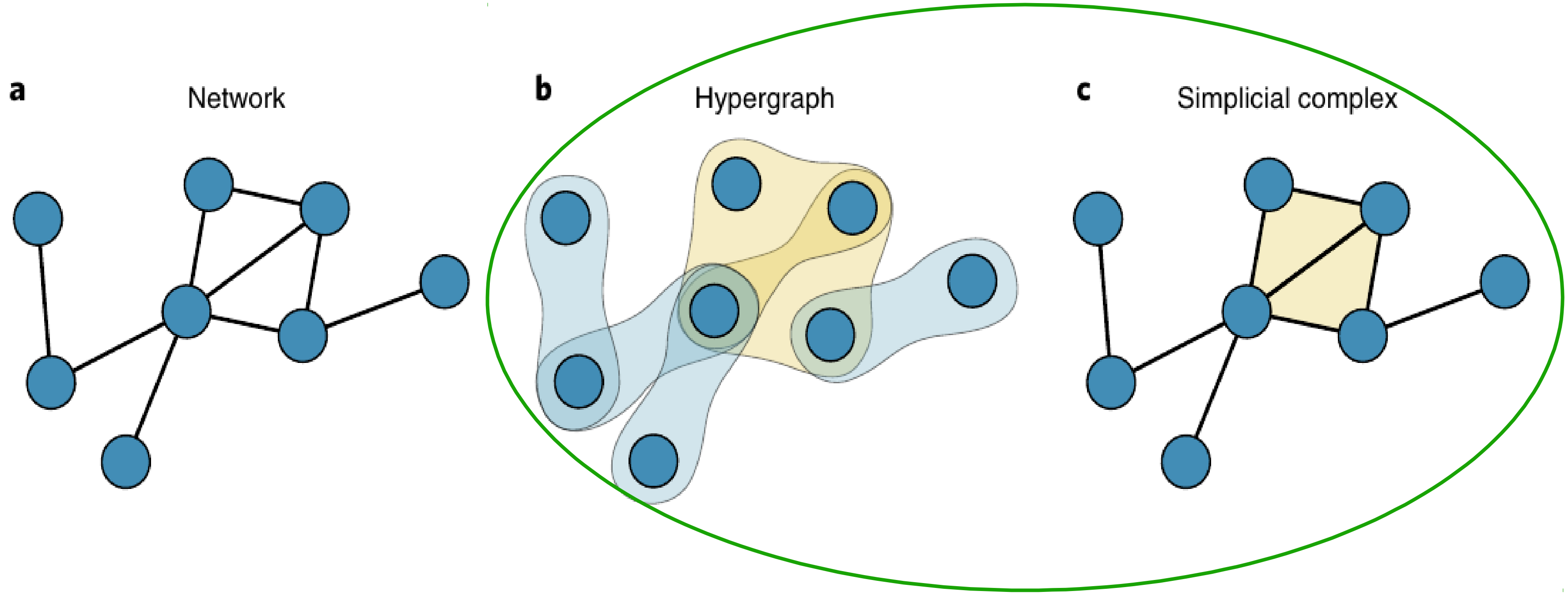



Battiston et al., Nat. Phys., 2021

# Higher-order Structures

adjacency matrix

adjacency tensors



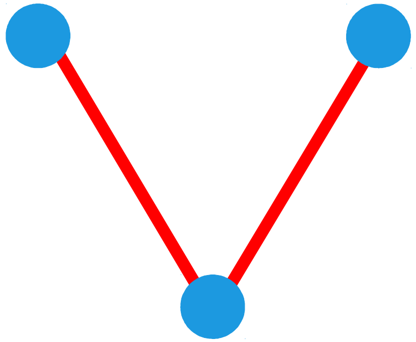

Battiston et al., Nat. Phys., 2021

# Higher-order nonlinear systems

$$\dot{\vec{x}}_i = \vec{f}(\vec{x}_i) + \sigma_1 \sum_j A_{ij}^{(1)} \vec{g}^{(1)}(\vec{x}_i, \vec{x}_j)$$

dynamics of $x_i$

pairwise coupling

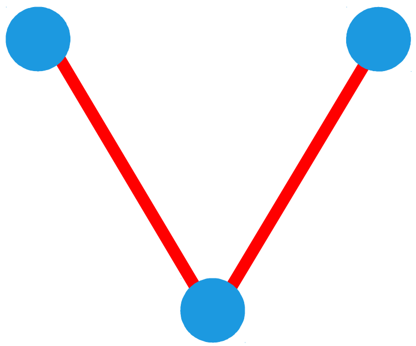


# Higher-order nonlinear systems

$$\dot{\vec{x}}_i = \vec{f}(\vec{x}_i) + \sigma_1 \sum_j A^{(1)}_{ij} \vec{g}^{(1)}(\vec{x}_i, \vec{x}_j) + \sigma_2 \sum_{j,k} A^{(2)}_{ijk} \vec{g}^{(2)}(\vec{x}_i, \vec{x}_j, \vec{x}_k)$$

dynamics of $x_i$

pairwise coupling



# Higher-order nonlinear systems

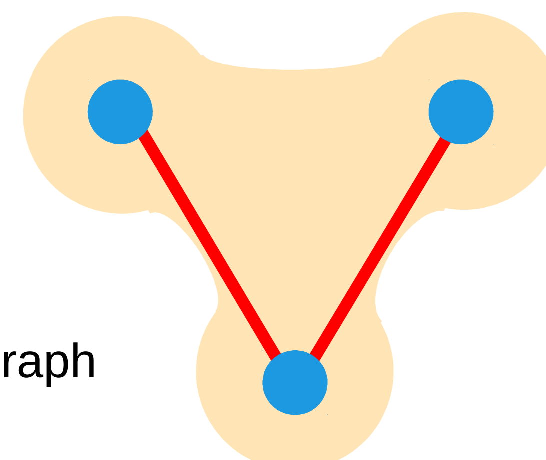
$$\dot{\vec{x}}_i = \vec{f}(\vec{x}_i) + \sigma_1 \sum_j A_{ij}^{(1)} \vec{g}^{(1)}(\vec{x}_i, \vec{x}_j) + \sigma_2 \sum_{j,k} A_{ijk}^{(2)} \vec{g}^{(2)}(\vec{x}_i, \vec{x}_j, \vec{x}_k)$$

dynamics of $x_i$

higher-order
(3-body) coupling

pairwise coupling

hypergraph



# Patterns in higher-order systems

$$
\begin{aligned}
\dot{u}_i &= f_1(u_i, v_i) + \sigma_1 D_u^{(1)} \sum_{j_1=1}^{N} A_{ij_1}^{(1)}\left(h_1^{(1)}(u_{j_1}) - h_1^{(1)}(u_i)\right) \\
&\quad + \sigma_2 D_u^{(2)} \sum_{j_1=1}^{N} \sum_{j_2=1}^{N} A_{ij_1j_2}^{(2)}\left(h_1^{(2)}(u_{j_1}, u_{j_2}) - h_1^{(2)}(u_i, u_i)\right) \\
\dot{v}_i &= f_2(u_i, v_i) + \sigma_1 D_v^{(1)} \sum_{j_1=1}^{N} A_{ij_1}^{(1)}\left(h_2^{(1)}(v_{j_1}) - h_2^{(1)}(v_i)\right) \\
&\quad + \sigma_2 D_v^{(2)} \sum_{j_1=1}^{N} \sum_{j_2=1}^{N} A_{ij_1j_2}^{(2)}\left(h_2^{(2)}(v_{j_1}, v_{j_2}) - h_2^{(2)}(v_i, v_i)\right)
\end{aligned}
$$

# Natural coupling

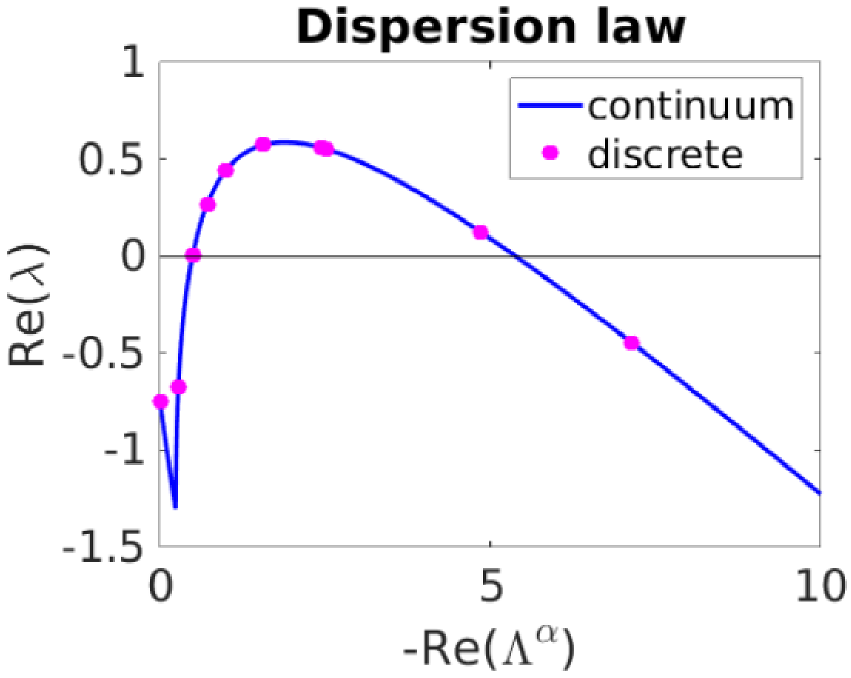
$$\frac{d}{dt}\vec{\xi} = \left(\mathbb{I}_N \otimes \mathbf{J}_0 + \sigma_1 \mathbf{L}^{(1)} \otimes \mathbf{J}_{H^{(1)}} + \sigma_2 \mathbf{L}^{(2)} \otimes \mathbf{J}_{H^{(2)}}\right)\vec{\xi}$$

$$\vec{h}^{(d)}(\vec{x}, \ldots, \vec{x}) = \ldots = \vec{h}^{(2)}(\vec{x}, \vec{x}) = \vec{h}^{(1)}(\vec{x})$$

same diffusion coefficients for every order

$$\frac{d}{dt}\vec{\xi} = \left[\mathbb{I}_N \otimes \mathbf{J}_0 + \left(\sigma_1 \mathbf{L}^{(1)} + \sigma_2 \mathbf{L}^{(2)}\right) \otimes \mathbf{J}_{H^{(1)}}\right]\vec{\xi}$$

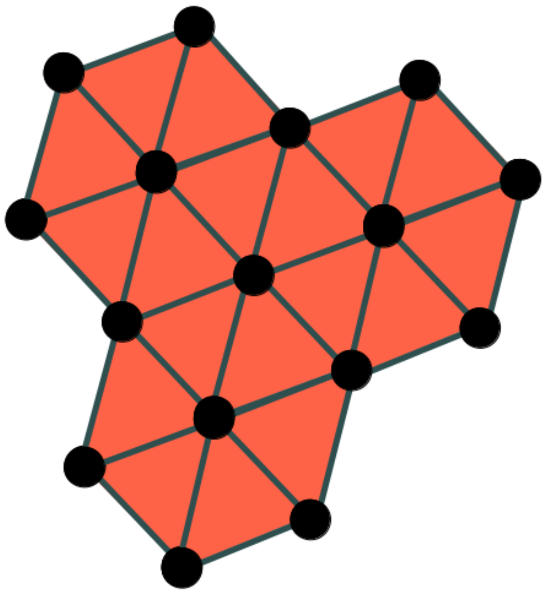
# Natural coupling




same diffusion coefficients
for every order

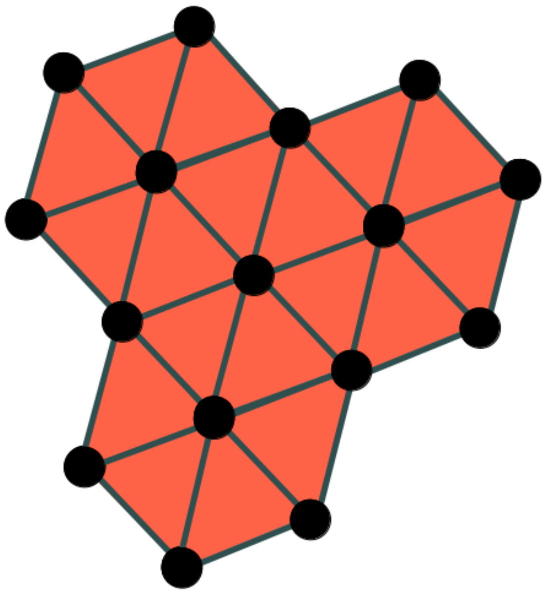
$$\frac{d}{dt}\vec{\xi} = \left[\mathbb{I}_N \otimes \mathbf{J}_0 + \left(\sigma_1 \mathbf{L}^{(1)} + \sigma_2 \mathbf{L}^{(2)}\right) \otimes \mathbf{J}_{H^{(1)}}\right]\vec{\xi}$$

# Regular topologies

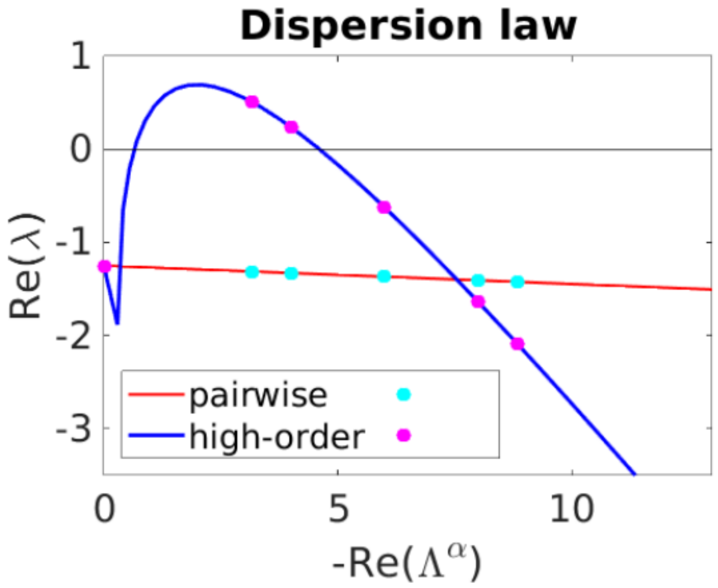


$$\mathbf{L}^{(2)} = 2\mathbf{L}^{(1)}$$

# Regular topologies

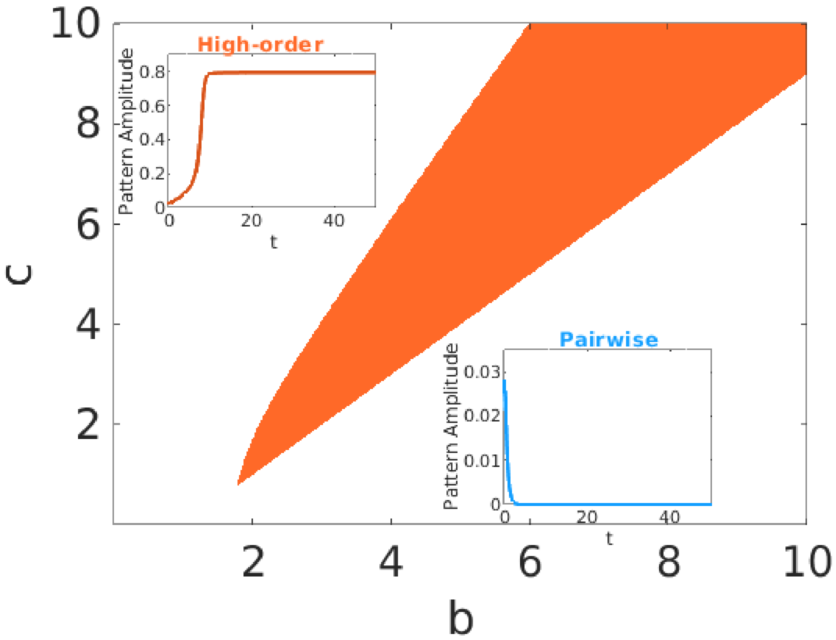


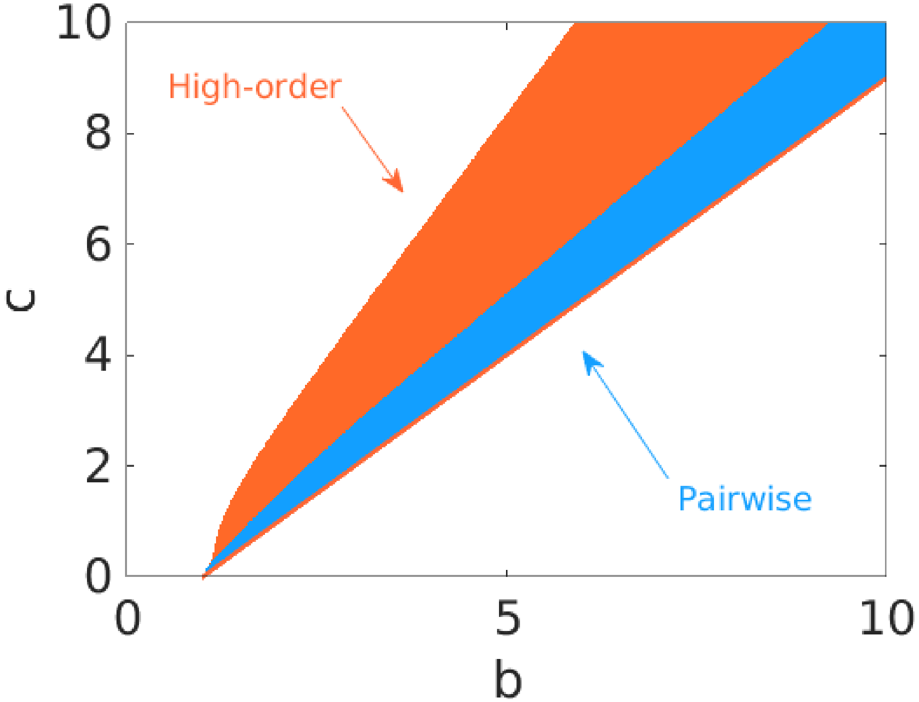
$$\mathbf{L}^{(2)} = 2\mathbf{L}^{(1)}$$



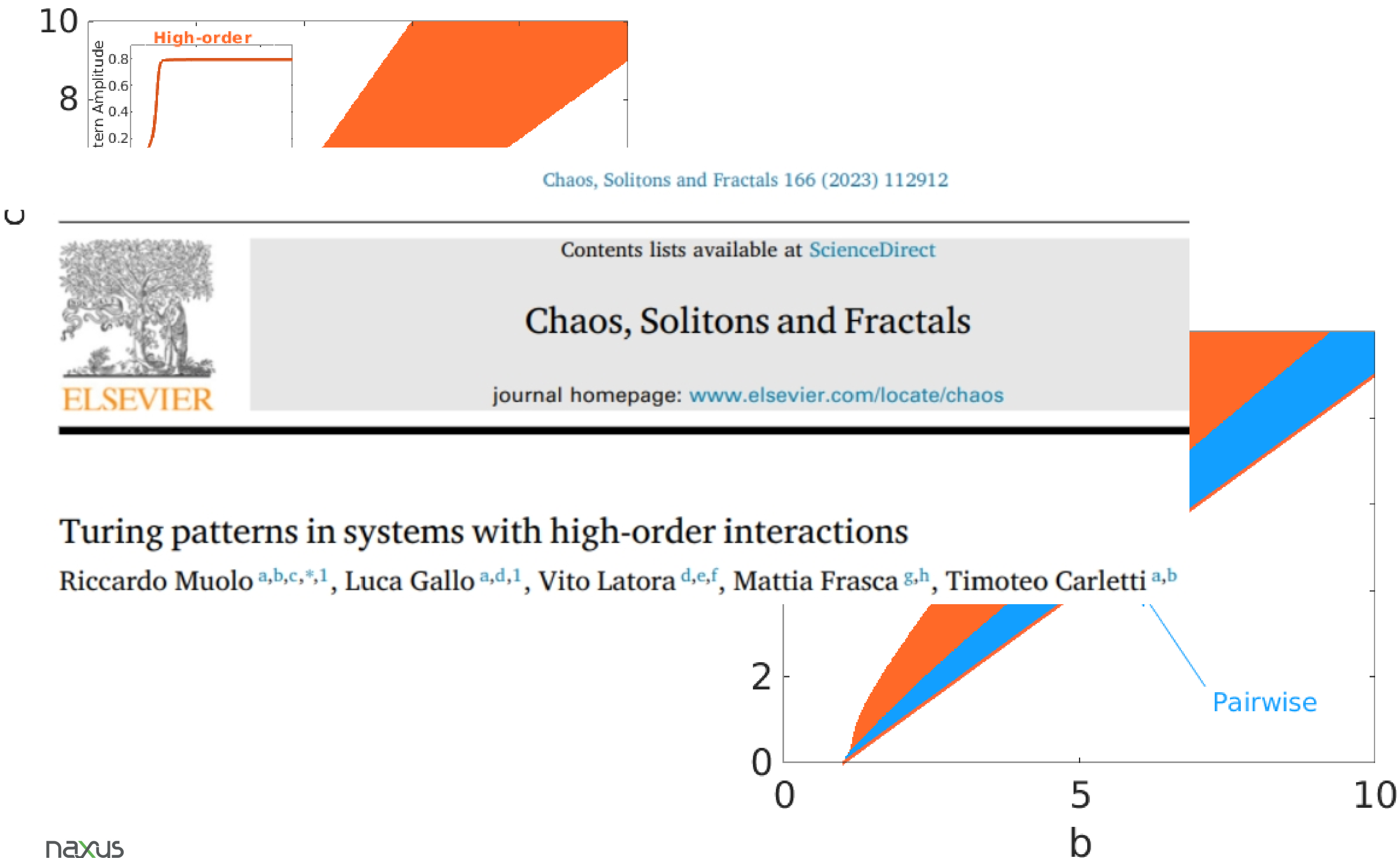

$$\frac{d}{dt}\vec{\xi} = \left(\mathbb{I}_N \otimes \mathbf{J}_0 + \mathbf{L}^{(1)} \otimes (\sigma_1 \mathbf{J}_{H^{(1)}} + 2\sigma_2 \mathbf{J}_{H^{(2)}})\right)\vec{\xi}$$

# General couplings



# General couplings





Chaos, Solitons and Fractals 166 (2023) 112912

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

journal homepage: www.elsevier.com/locate/chaos


ELSEVIER

## Turing patterns in systems with high-order interactions


Riccardo Muolo [a,b,c,*,1], Luca Gallo [a,d,1], Vito Latora [d,e,f], Mattia Frasca [g,h], Timoteo Carletti [a,b]

# Higher-order nonlinear systems

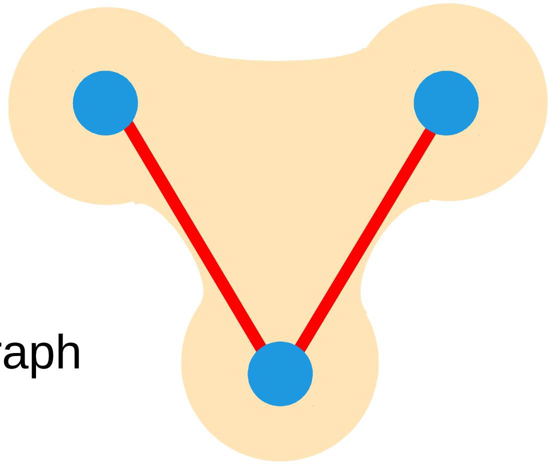
$$\dot{\vec{x}}_i = \vec{f}(\vec{x}_i) + \sigma_1 \sum_j A^{(1)}_{ij}\vec{g}^{(1)}(\vec{x}_i,\vec{x}_j) + \sigma_2 \sum_{j,k} A^{(2)}_{ijk}\vec{g}^{(2)}(\vec{x}_i,\vec{x}_j,\vec{x}_k)$$

dynamics of $x_i$

higher-order
(3-body) coupling

pairwise coupling

hypergraph



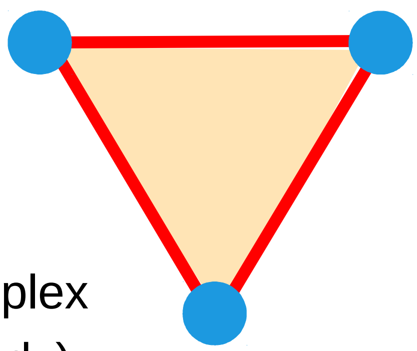
# Higher-order nonlinear systems

$$\dot{\vec{x}}_i = \vec{f}(\vec{x}_i) + \sigma_1 \sum_j A^{(1)}_{ij} \vec{g}^{(1)}(\vec{x}_i, \vec{x}_j) + \sigma_2 \sum_{j,k} A^{(2)}_{ijk} \vec{g}^{(2)}(\vec{x}_i, \vec{x}_j, \vec{x}_k)$$

dynamics of $x_i$

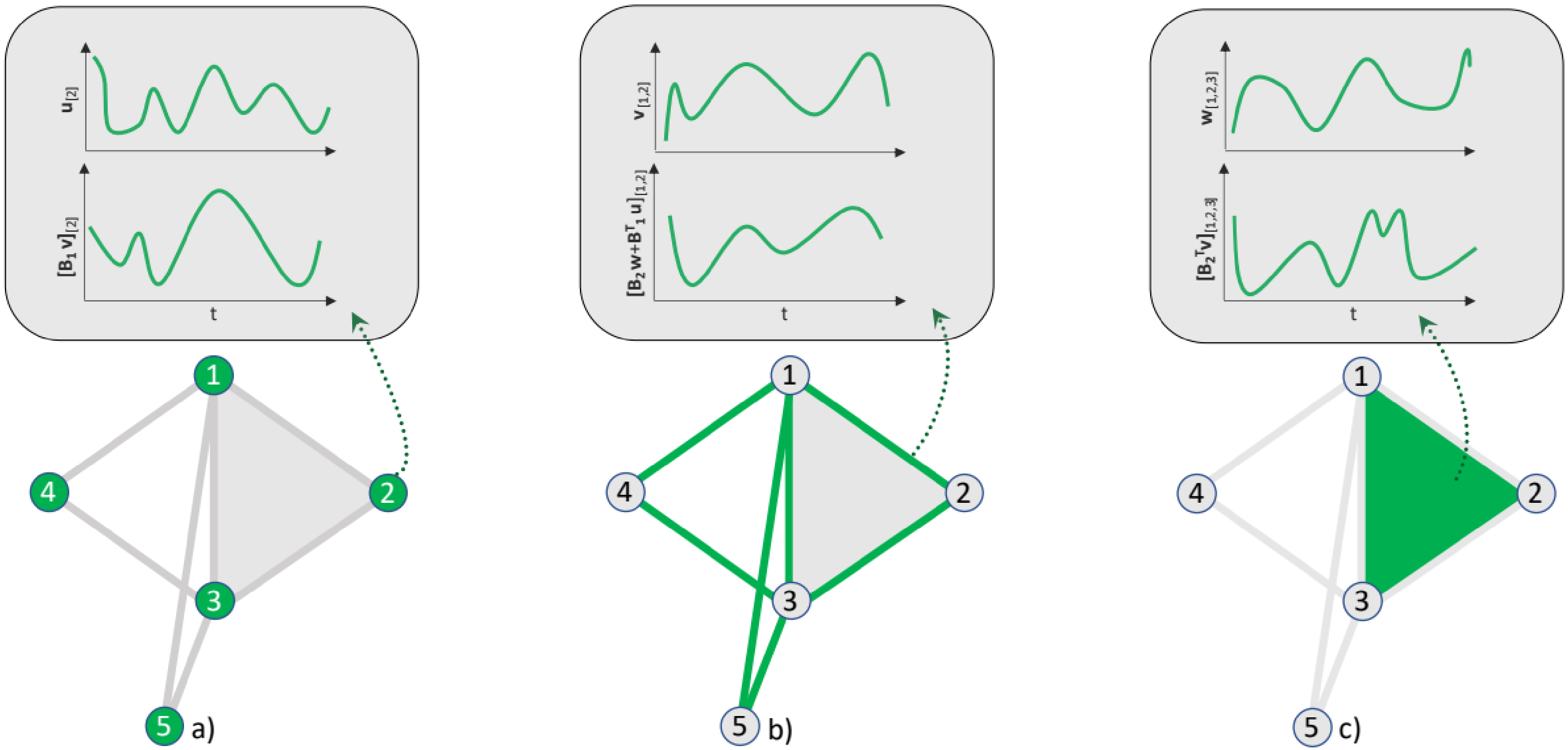
higher-order
(3-body) coupling

pairwise coupling

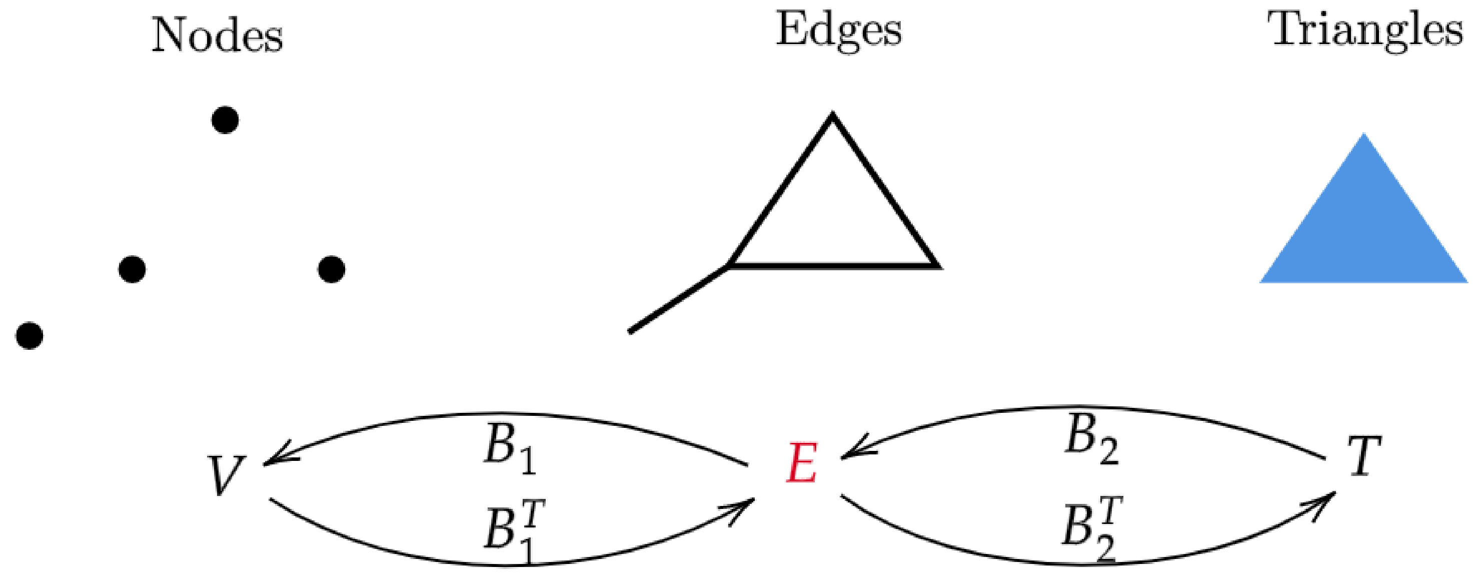


simplicial complex
(still many-body)

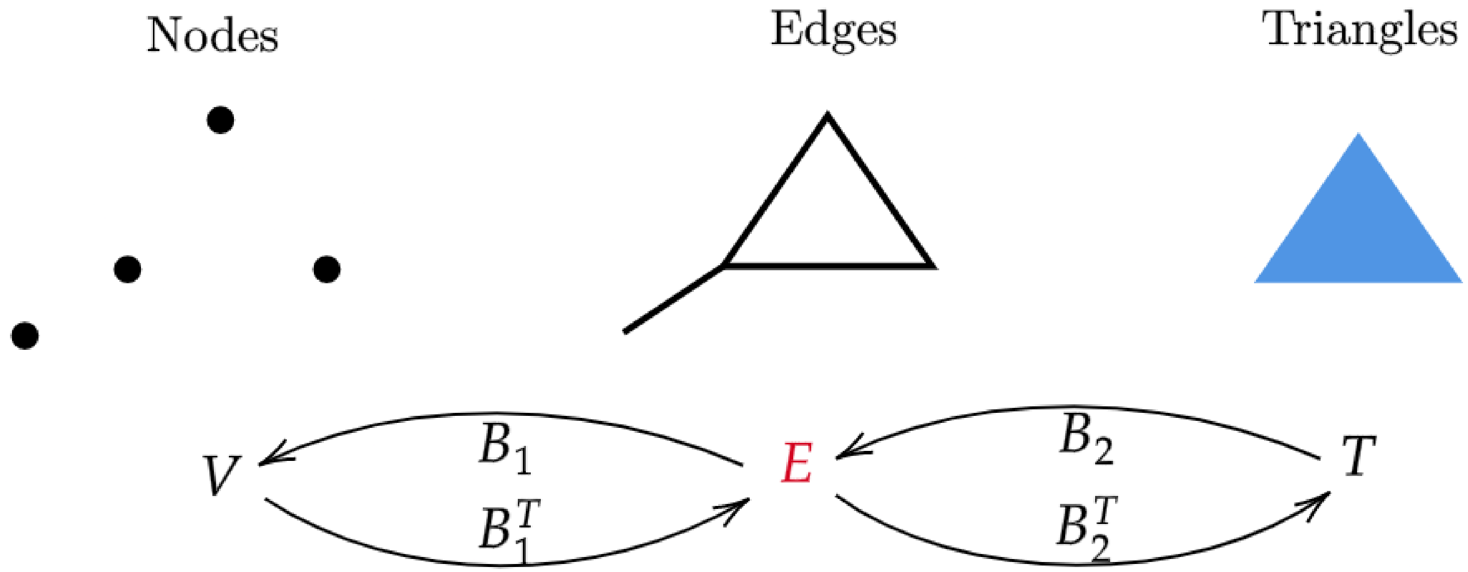
# Topological Signals



# Boundary Operators



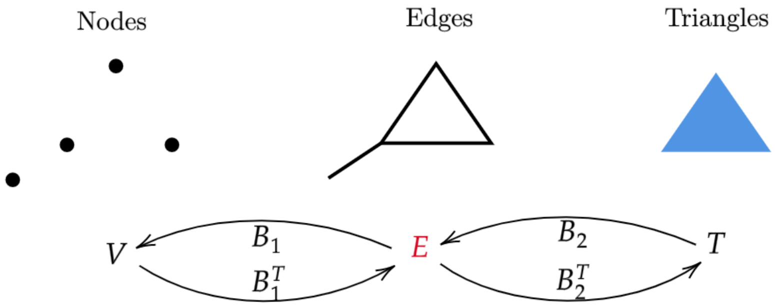
# Boundary Operators




$$\mathbf{L}_0 = \mathbf{B}_1 \mathbf{B}_1^\top$$

# Hodge Laplacians

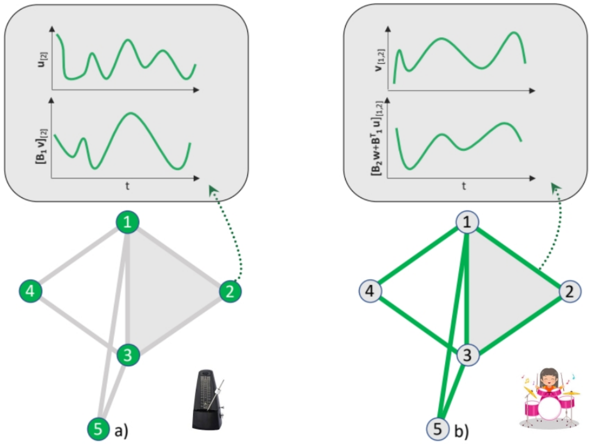
$$\mathbf{L}_0 = \mathbf{B}_1 \mathbf{B}_1^\top$$




$$\mathbf{L}_1 = \mathbf{B}_1^\top \mathbf{B}_1 + \mathbf{B}_2 \mathbf{B}_2^\top$$

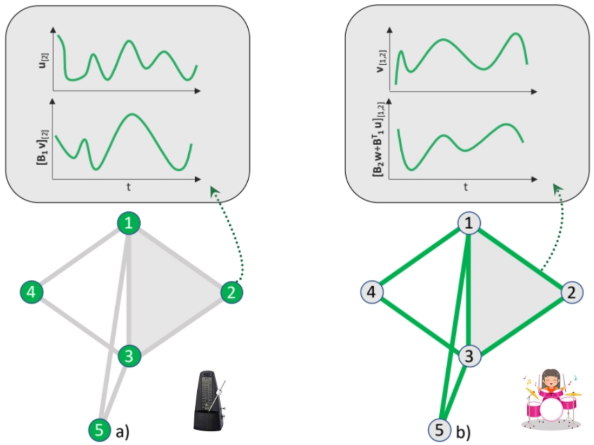
$$\mathbf{L}_2 = \mathbf{B}_2^\top \mathbf{B}_2$$

# Topological reaction-diffusion



# Topological reaction-diffusion

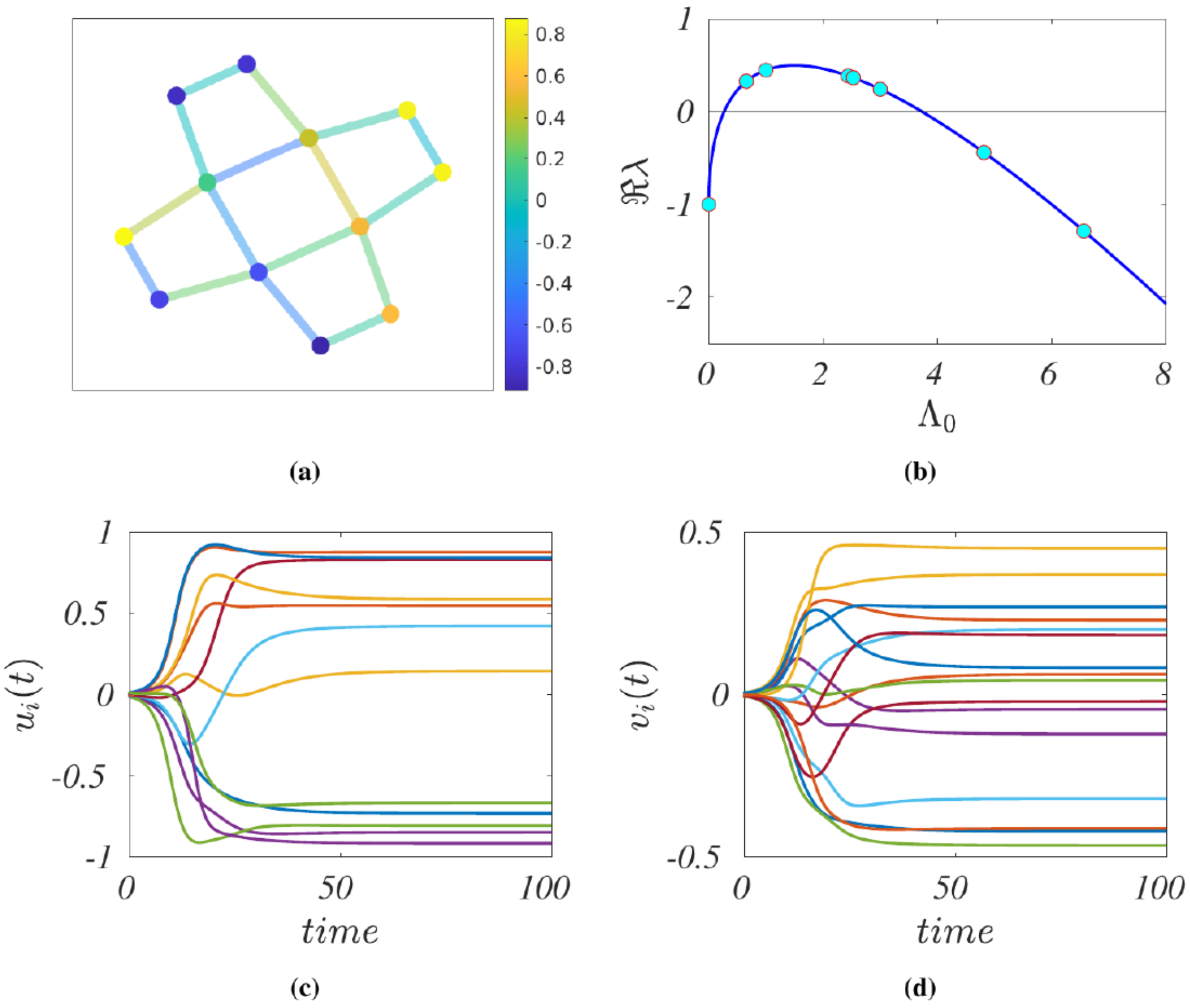
$$\dot{\Phi} = F(\Phi, \mathcal{D}\Phi) - \gamma \mathcal{L}\Phi$$




$$\frac{du}{dt} = f(u, \mathbf{B}_1 v) - D_0 \, \mathbf{L}_0 \, u,$$

$$\frac{dv}{dt} = g\left(v, \mathbf{B}_1^\top u\right) - D_1 \, \mathbf{L}_1 \, v$$

# Higher-order patterns




naxys Namur Institute for Complex Systems

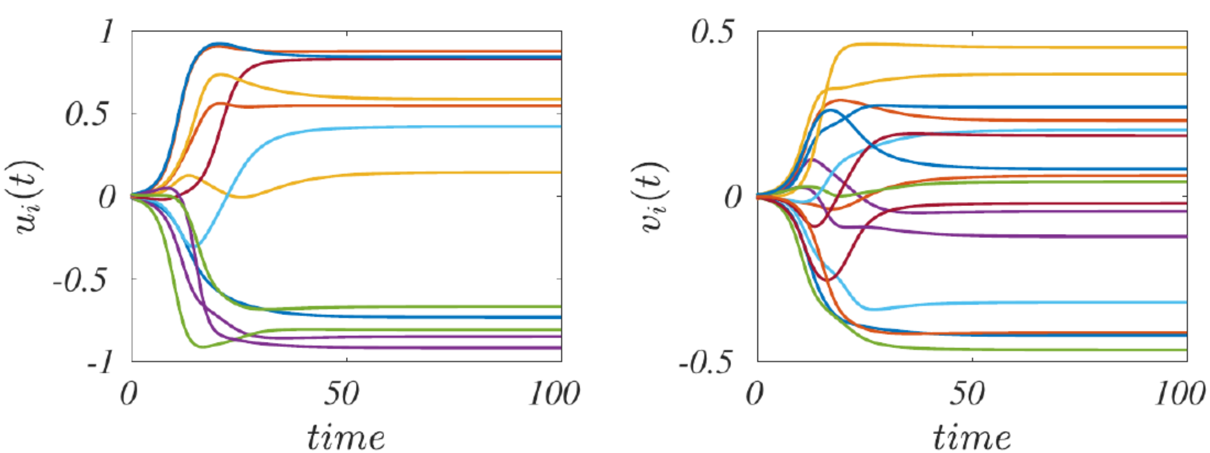
# Higher-order patterns


PHYSICAL REVIEW E **106**, 064314 (2022)

**Diffusion-driven instability of topological signals coupled by the Dirac operator**

Lorenzo Giambagli,[1,2,*] Lucille Calmon,[3,†] Riccardo Muolo,[2,4,†] Timoteo Carletti,[2] and Ginestra Bianconi[3,5,‡]

**(a)** **(b)**




**(c)** **(d)**

# Acknowledgements


Mattia Frasca & Valentina Gambuzza
(University of Catania)

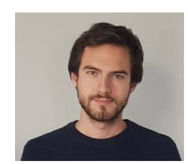
Luca Gallo (Catania, now at CEU Vienna)

Lucille Calmon
(Queen Mary, now Paris)

Teo Carletti (University of Namur)

Vito Latora (Queen Mary & Catania)

Ginestra Bianconi
(Queen Mary)

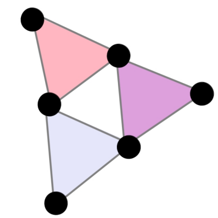


Lorenzo Giambagli
(University of Florence & Namur)

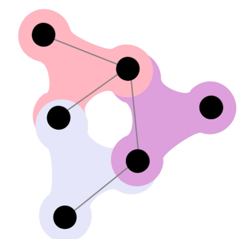
# Thank you for your attention

**Questions?**

## Take Home Message



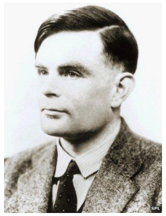
We extended the formalism of high-order interactions
to reaction-diffusion systems yielding Turing patterns



The effects of the many-body dynamics may
enhance or hamper the formation of patterns

Through the tools of algebraic topology
the theory can be further extended
to topological signals

naxys
Namur Institute
for Complex Systems

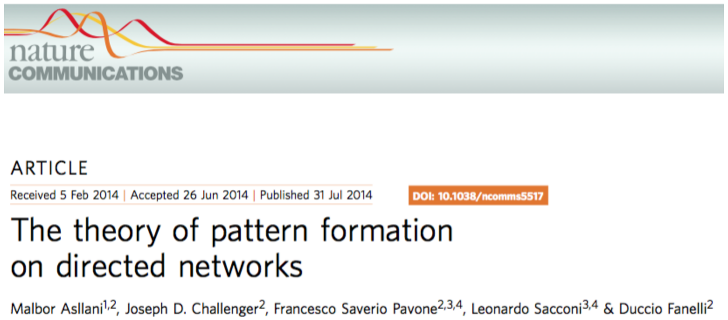


UNIVERSITÉ DE NAMUR

# BONUS SLIDES

## Effects of non-normality on Turing patterns and synchronization

# Directed Networks


nature COMMUNICATIONS

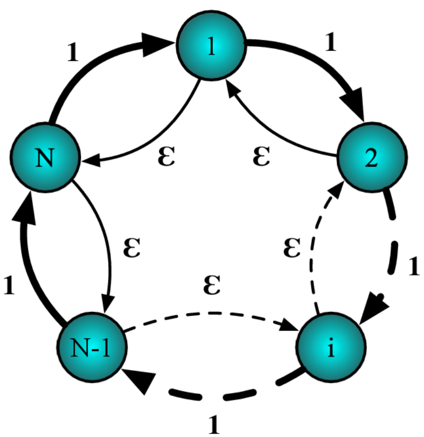
ARTICLE

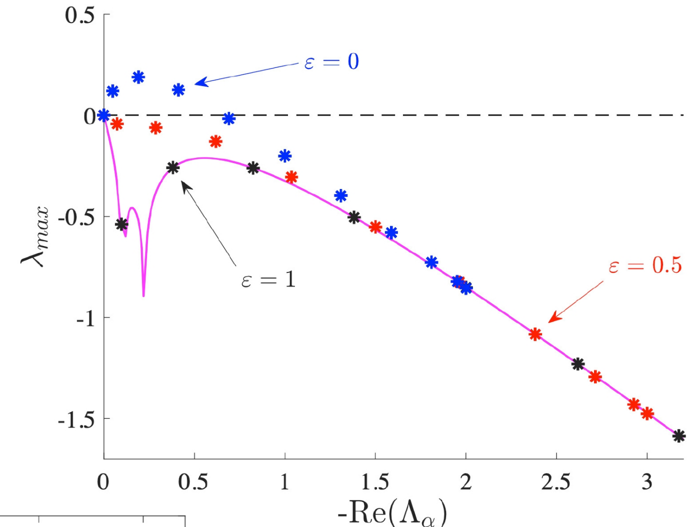
Received 5 Feb 2014 | Accepted 26 Jun 2014 | Published 31 Jul 2014

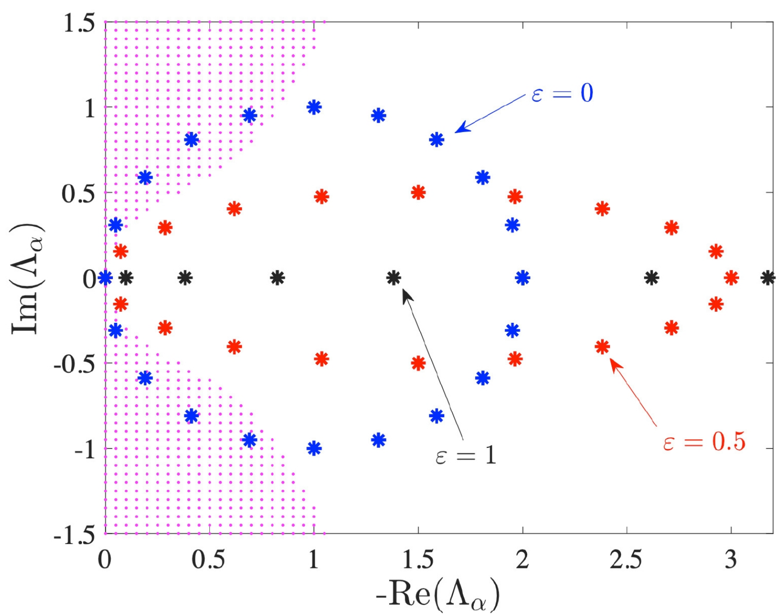
DOI: 10.1038/ncomms5517

The theory of pattern formation on directed networks

Malbor Asllani[1,2], Joseph D. Challenger[2], Francesco Saverio Pavone[2,3,4], Leonardo Sacconi[3,4] & Duccio Fanelli[2]



# Non-normal Networks

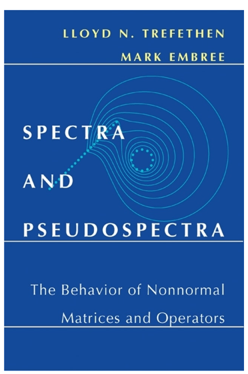
A network is non-normal when $A^{*}A \neq AA^{*}$

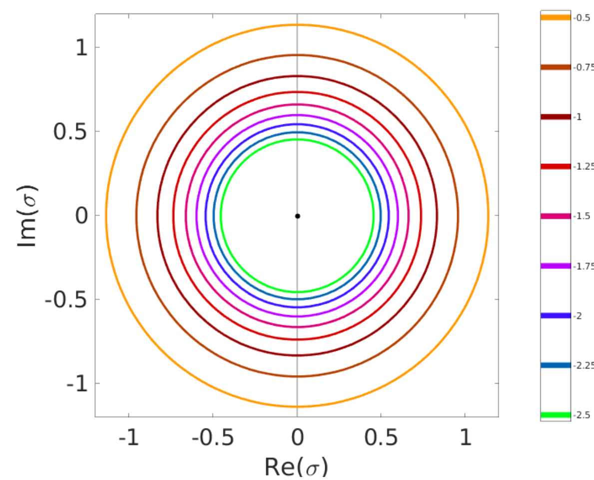
RESEARCH ARTICLE | NETWORK SCIENCE

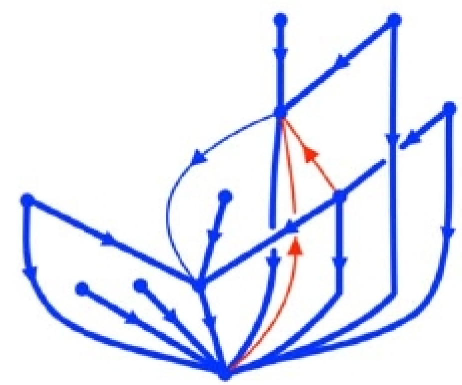
## Structure and dynamical behavior of non-normal networks

Malbor Asllani[1,2], Renaud Lambiotte[1] and Timoteo Carletti[2,*]
+ See all authors and affiliations

*Science Advances* 12 Dec 2018:
Vol. 4, no. 12, eaau9403
DOI: 10.1126/sciadv.aau9403



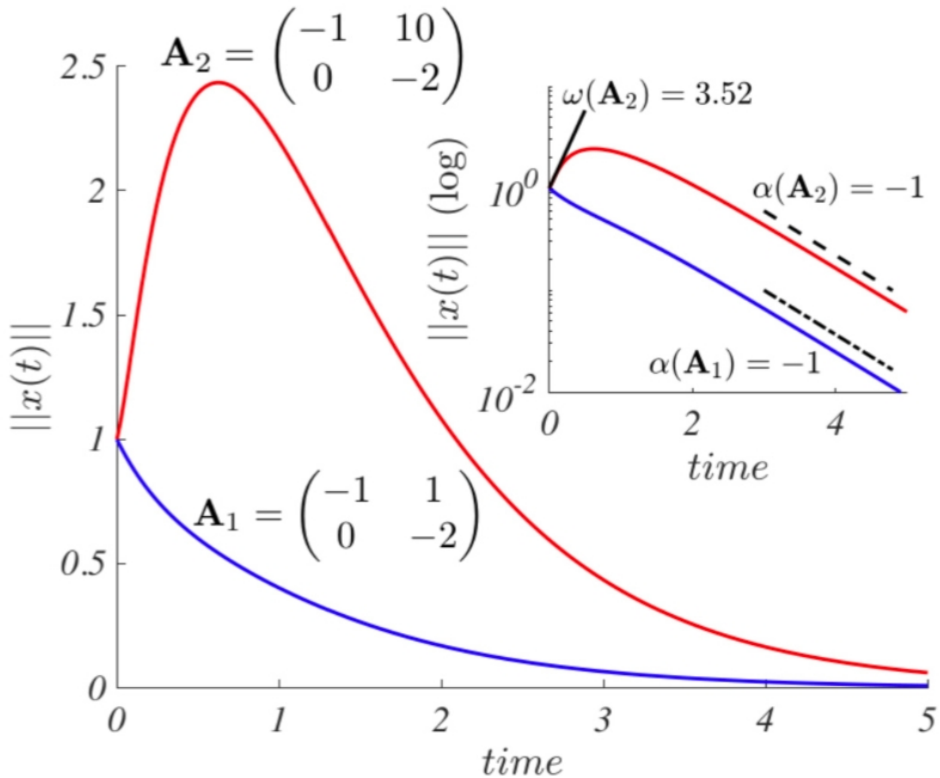



$$\beta\text{-pseudospectrum } \lambda_{\beta}(A) \rightarrow \lambda(A+B) \; ||B|| < \beta$$

# Effects on the dynamics

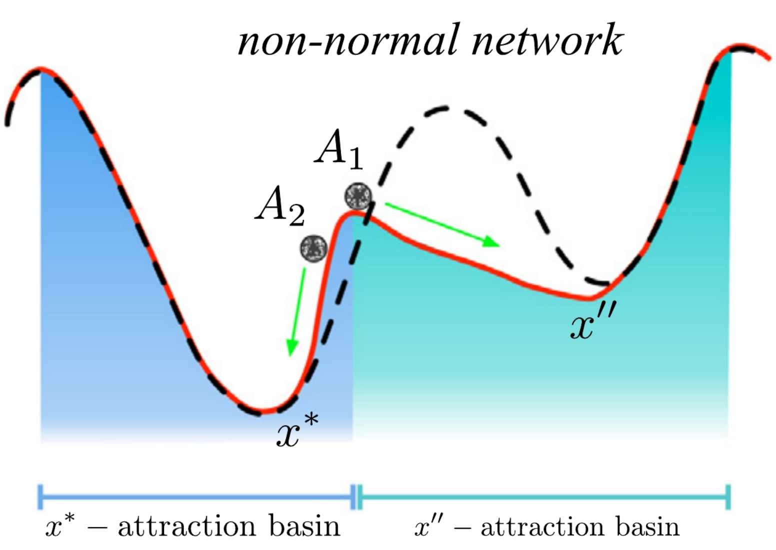
ω largest eigenvalue of the hermitian (simmetric) part

Linear dynamics

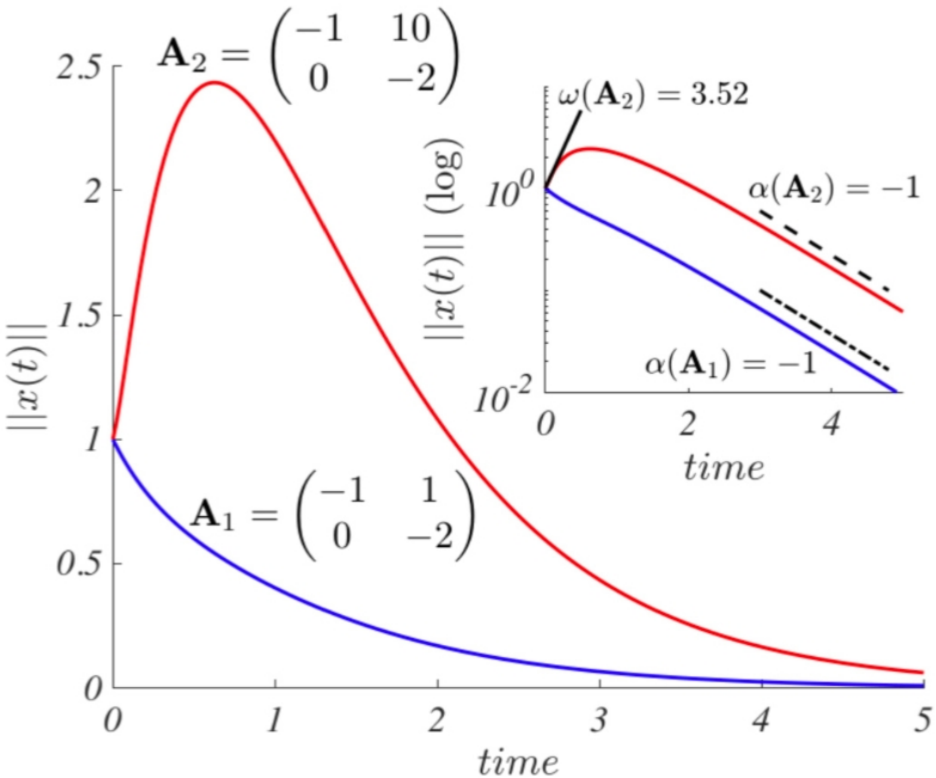


# Effects on the dynamics

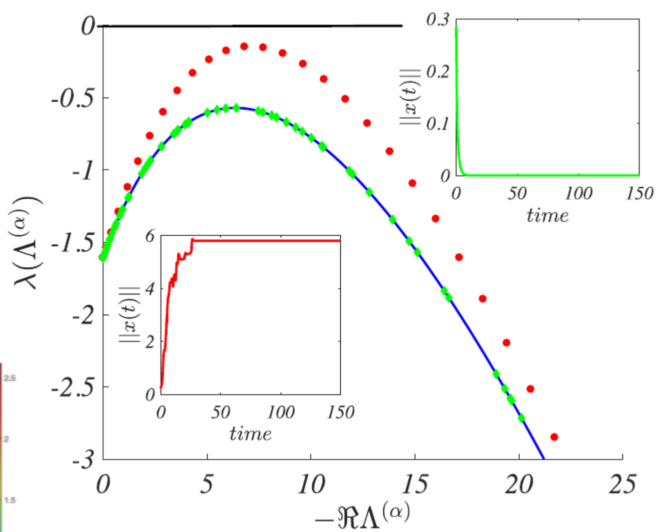
Non-linear dynamics

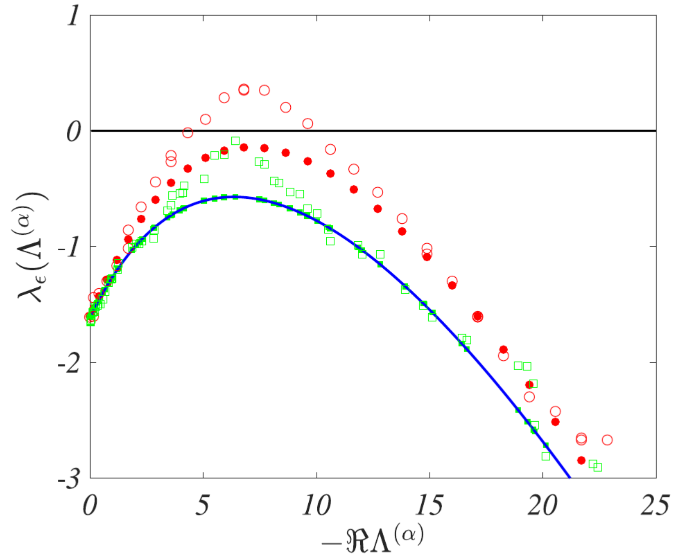


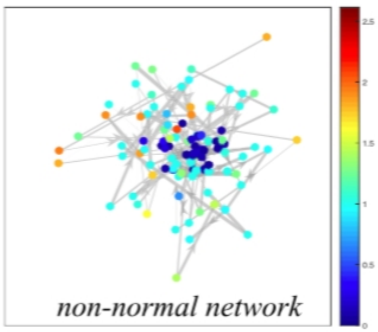

Linear dynamics

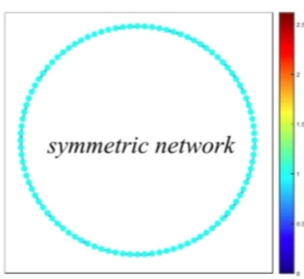


# Patterns of Non-normality





Journal of Theoretical Biology 480 (2019) 81–91

Contents lists available at ScienceDirect

Journal of Theoretical Biology

journal homepage: www.elsevier.com/locate/jtb


Patterns of non-normality in networked systems


Riccardo Muolo[a], Malbor Asllani[b,c,*], Duccio Fanelli[d], Philip K. Maini[b], Timoteo Carletti[e]

# Synchronization and non-normality

entropy

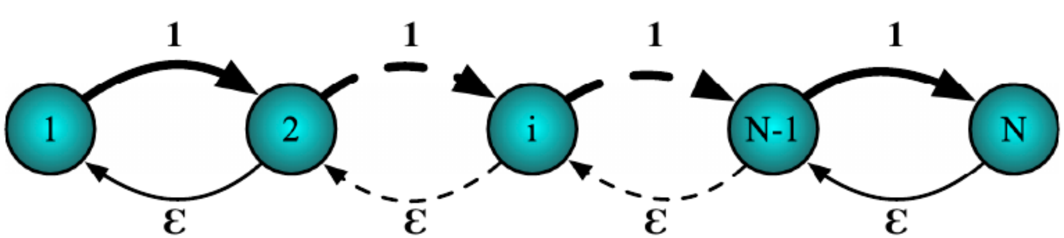
*Entropy* **2021**, *23*, 36.

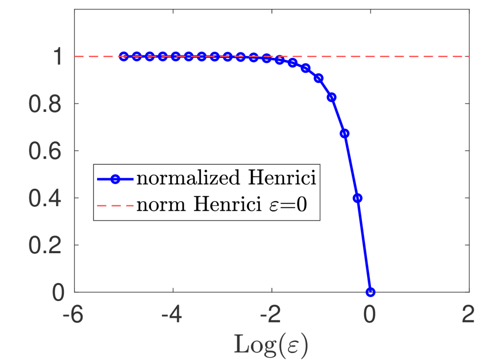
MDPI

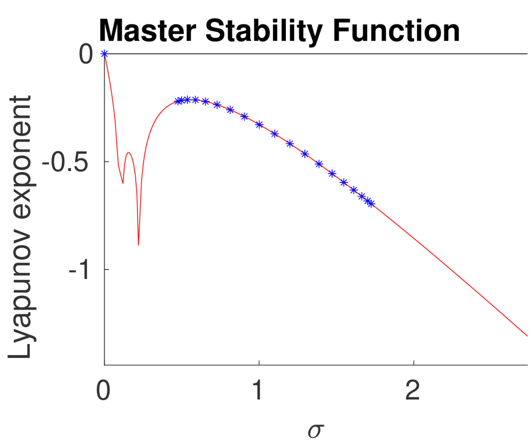
*Article*

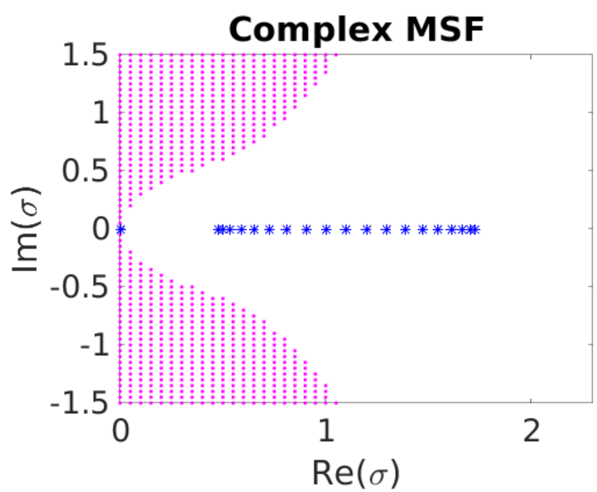
**Synchronization Dynamics in Non-Normal Networks: The Trade-Off for Optimality**

Riccardo Muolo [1,*], Timoteo Carletti [1], James P. Gleeson [2] and Malbor Asllani [1,2]



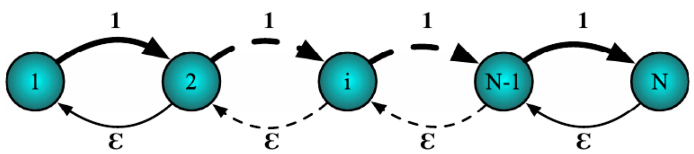



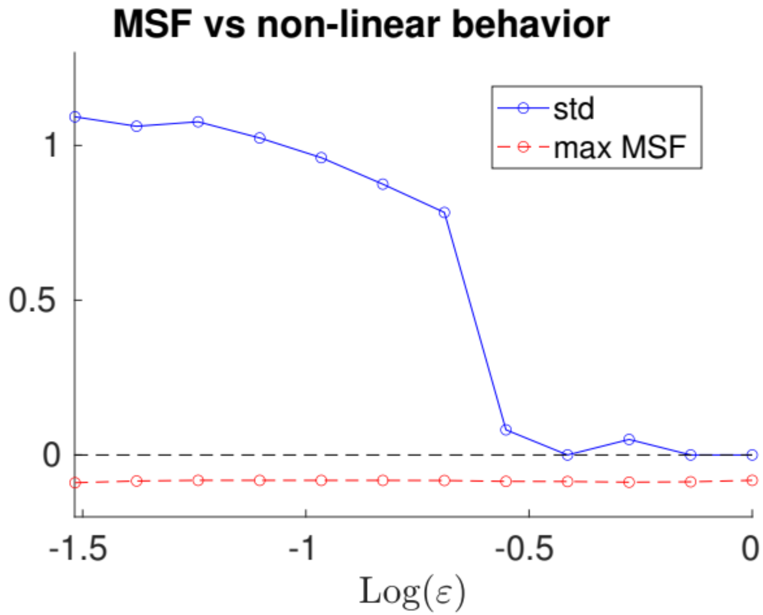



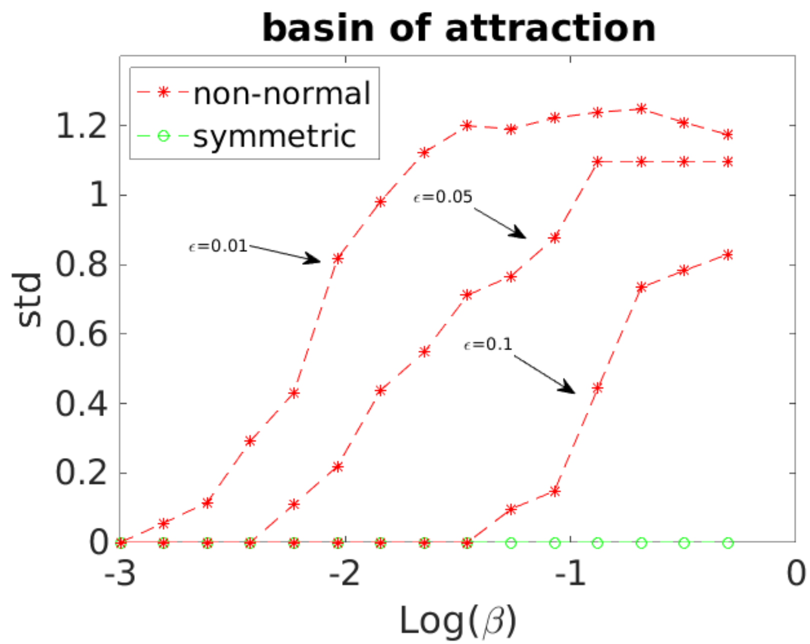



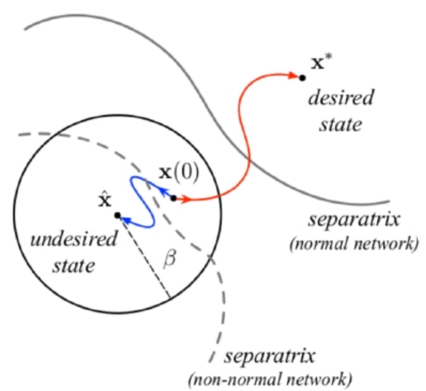

naxys
Namur Institute for Complex Systems

# De-synchronization



Namur Institute for Complex Systems

# BONUS SLIDES

## Effects of directed higher-order interactions on synchronization

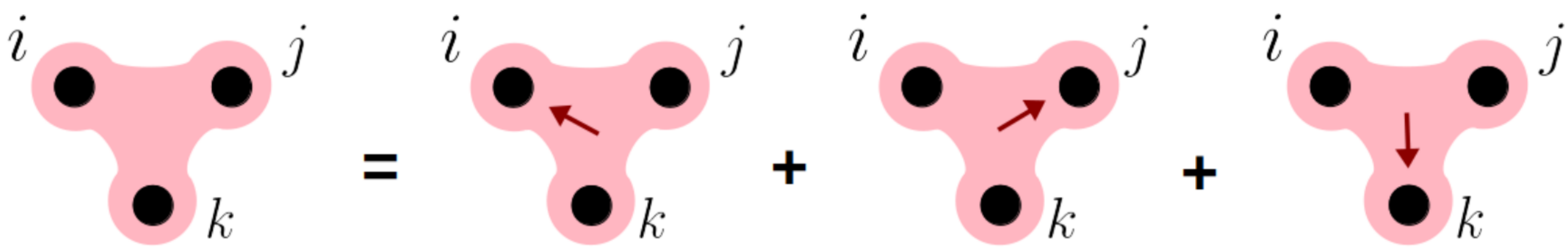
東京工業大学 Tokyo Institute of Technology

UNIVERSITÉ DE NAMUR

# What about directionality?

certain interactions are naturally
high-order and asymmetric,
e.g., peer pressure,
chemical reactions, etc

directed hypergraphs exist but
are used to study information flow
and not dynamics on them




$A^{(2)}_{\pi(ijk)} = 1$

$A^{(2)}_{i\pi(jk)} = 1$ $A^{(2)}_{j\pi(ik)} = 0$ $A^{(2)}_{k\pi(ij)} = 0$

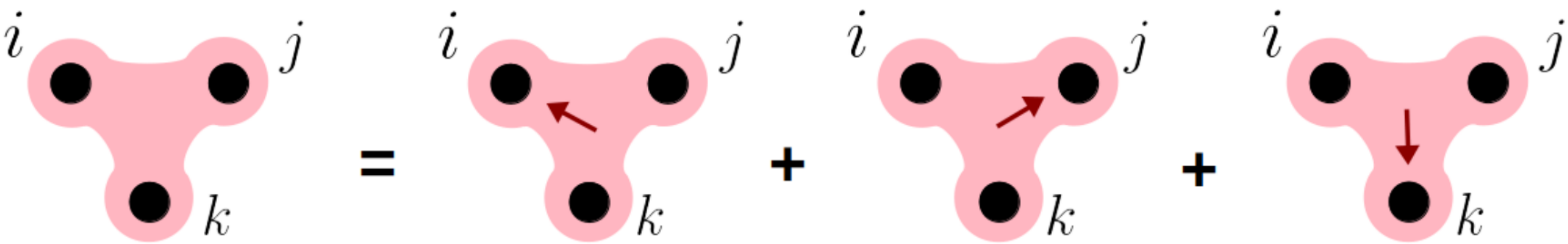
$A^{(2)}_{j\pi(ik)} = 1$ $A^{(2)}_{i\pi(jk)} = 0$ $A^{(2)}_{k\pi(ij)} = 0$

$A^{(2)}_{k\pi(ij)} = 1$ $A^{(2)}_{j\pi(ik)} = 0$ $A^{(2)}_{i\pi(jk)} = 0$

# What about directionality?

certain interactions are naturally high-order and asymmetric e.g., peer pressure, chemical reactions, etc

directed hypergraphs exist but are used to study information flow and not dynamics on them




$A^{(2)}_{\pi(ijk)} = 1$

$A^{(2)}_{i\pi(jk)} = 1$ $\quad A^{(2)}_{j\pi(ik)} = 0$ $\quad A^{(2)}_{k\pi(ij)} = 0$

$A^{(2)}_{j\pi(ik)} = 1$ $\quad A^{(2)}_{i\pi(jk)} = 0$ $\quad A^{(2)}_{k\pi(ij)} = 0$

$A^{(2)}_{k\pi(ij)} = 1$ $\quad A^{(2)}_{j\pi(ik)} = 0$ $\quad A^{(2)}_{i\pi(jk)} = 0$

NOVELTY : elementary decomposition of undirected hyperedges + tensor formalism

# What about directionality?

certain interactions are naturally high-order and asymmetric
e.g., peer pressure, chemical reactions, etc

directed hypergraphs exist but are used to study information flow and not dynamics on them

**communications**
physics

ARTICLE

Check for updates

https://doi.org/10.1038/s42005-022-01040-9 OPEN

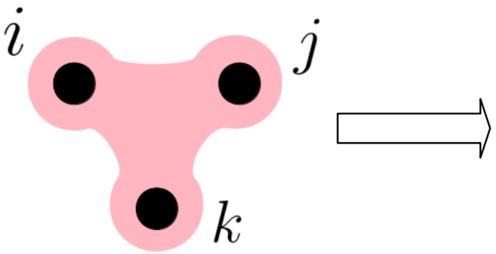
## Synchronization induced by directed higher-order interactions

Luca Gallo [1,2,3,10]✉, Riccardo Muolo [3,4,5,10], Lucia Valentina Gambuzza[6], Vito Latora [1,2,7,8], Mattia Frasca[6,9] & Timoteo Carletti [3,4]

NOVELTY : elementary decomposition of undirected hyperedges + tensor formalism

# 1-directed hypergraphs

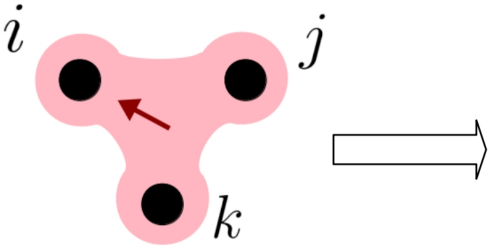
*undirected 2-hyperedge*




$$\begin{aligned} \dot{x}_i &\sim g^{(2)}(x_i, x_j, x_k) \\ \dot{x}_j &\sim g^{(2)}(x_j, x_i, x_k) \\ \dot{x}_k &\sim g^{(2)}(x_k, x_j, x_i) \end{aligned}$$

$$A^{(d)}_{ij_1\ldots j_d} = 1 \Rightarrow A^{(d)}_{i\pi(j_1\ldots j_d)} = 1$$

*1-directed 2-hyperedge*




$$\begin{aligned} \dot{x}_i &\sim g^{(2)}(x_i, x_j, x_k) \\ \dot{x}_j &\sim 0 \\ \dot{x}_k &\sim 0 \end{aligned}$$

$$A^{(d)}_{\pi(i_1,\ldots,i_m)\pi'(j_1,\ldots,j_s)} = 1$$

M-directed hypergraphs

# Global coupling matrix

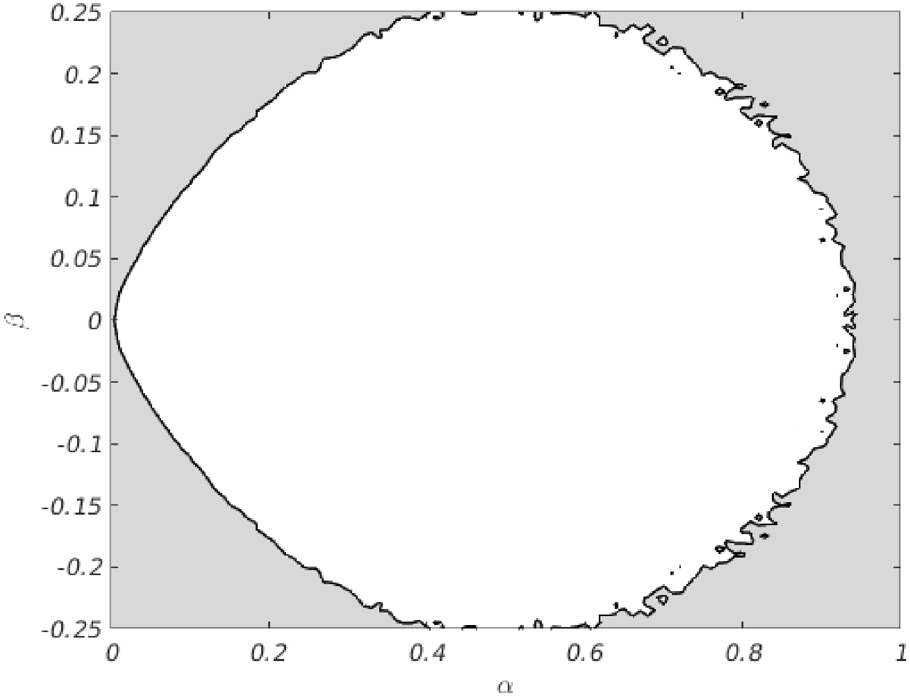
$$\begin{cases} \dot{x}_i = -y_i - z_i + \sigma_1 \sum_{j=1}^{N} A_{ij}^{(1)}(x_j^3 - x_i^3) + \sigma_2 \sum_{j,k=1}^{N} A_{ijk}^{(2)}(x_j^2 x_k - x_i^3) \\ \dot{y}_i = x_i + a y_i \\ \dot{z}_i = b + z_i(x_i - c) \end{cases}$$

Rössler with
cubic x-x coupling

$$\delta\dot{\vec{x}} = \left[\mathbb{I}_N \otimes JF - \mathcal{M} \otimes JH\right]\delta\vec{x}$$

$$\mathcal{M} = \sigma_1 \mathbf{L}^{(1)} + \sigma_2 \mathbf{L}^{(2)}$$

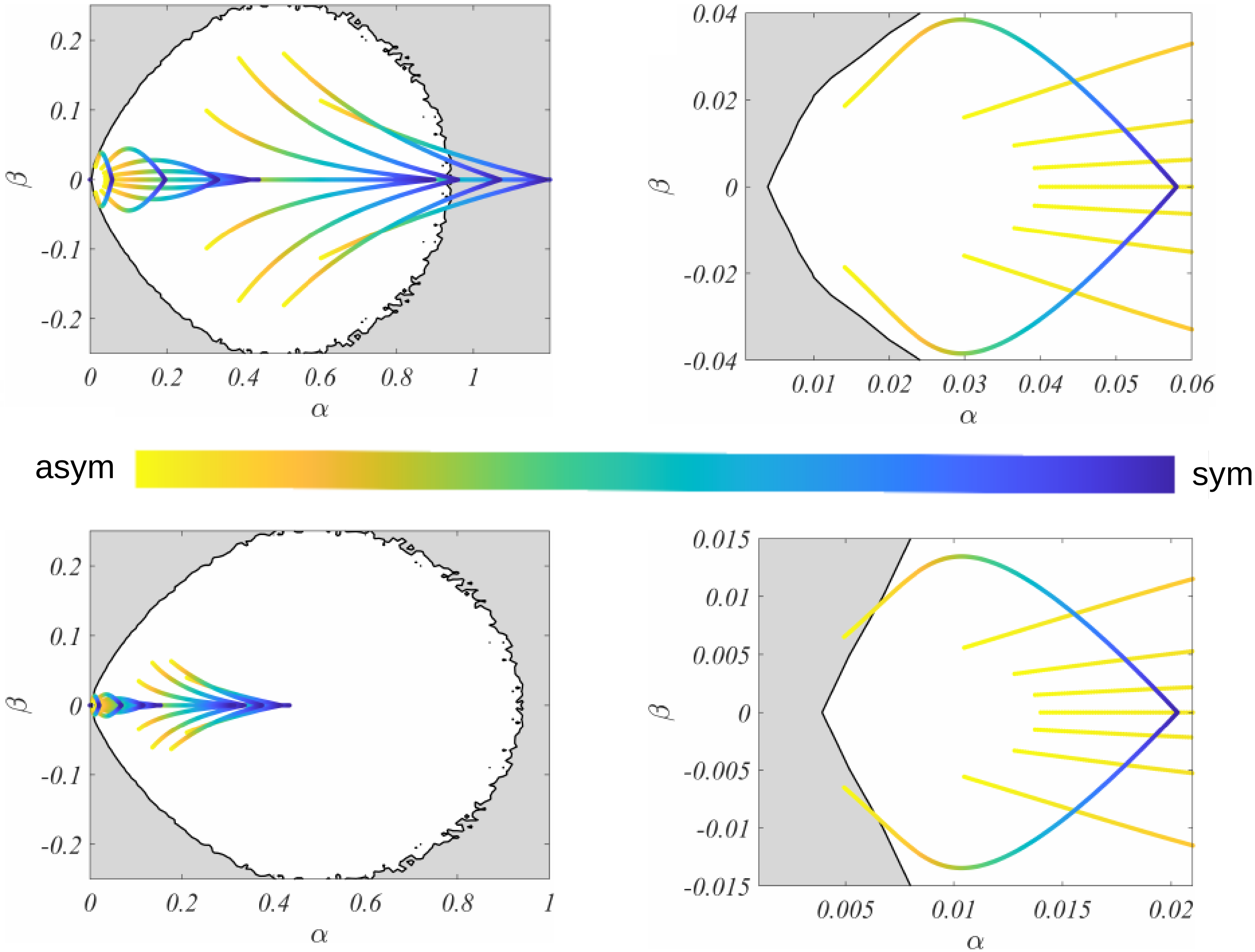
$$\dot{\vec{\xi}} = \left[JF(\vec{x}^s) - (\alpha + i\beta)JH(\vec{x}^s)\right]\vec{\xi}$$



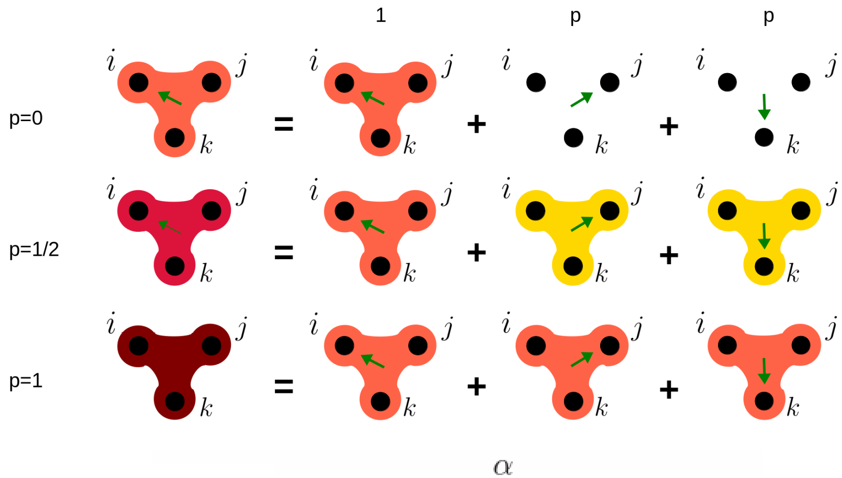
# Effects of topology

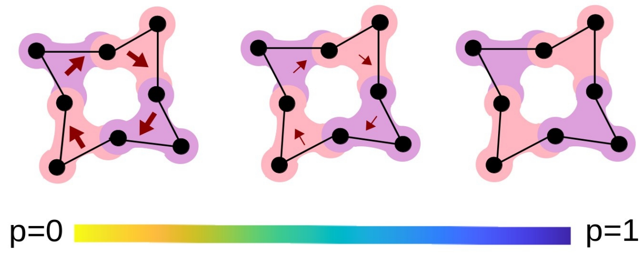
eigenvalues of

$\mathcal{M}$

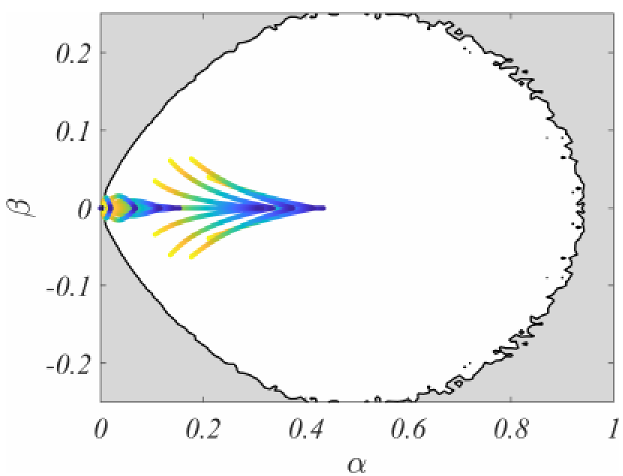


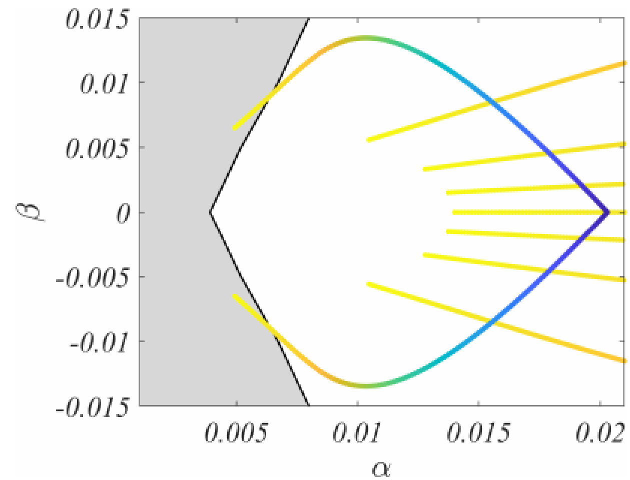
# Effects of topology







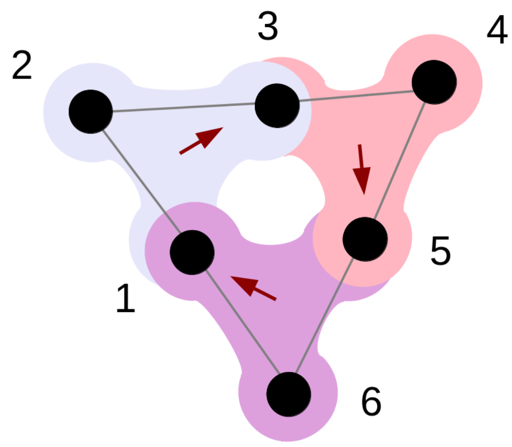



# 1-directed hypergraphs

$$A^{(2)}(p) = (\{A^{(2)}_{1jk}\}, \ldots, \{A^{(2)}_{6jk}\}) =$$

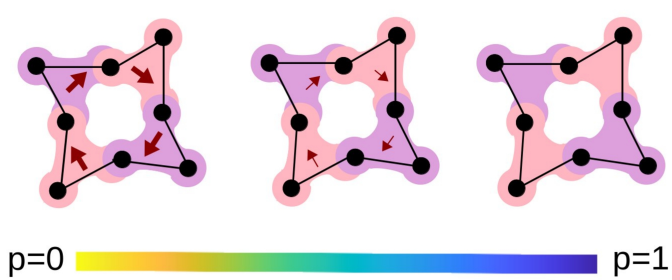
$$\left(\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&p&0&0&0\\ 0&p&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&p&0&0&0\\ 0&0&0&0&0&0\\ p&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&p&0\\ 0&0&0&p&0&0\\ 0&0&0&0&0&0 \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&p&0\\ 0&0&0&0&0&0\\ 0&0&p&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&p\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ p&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&p&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ p&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}\right)$$

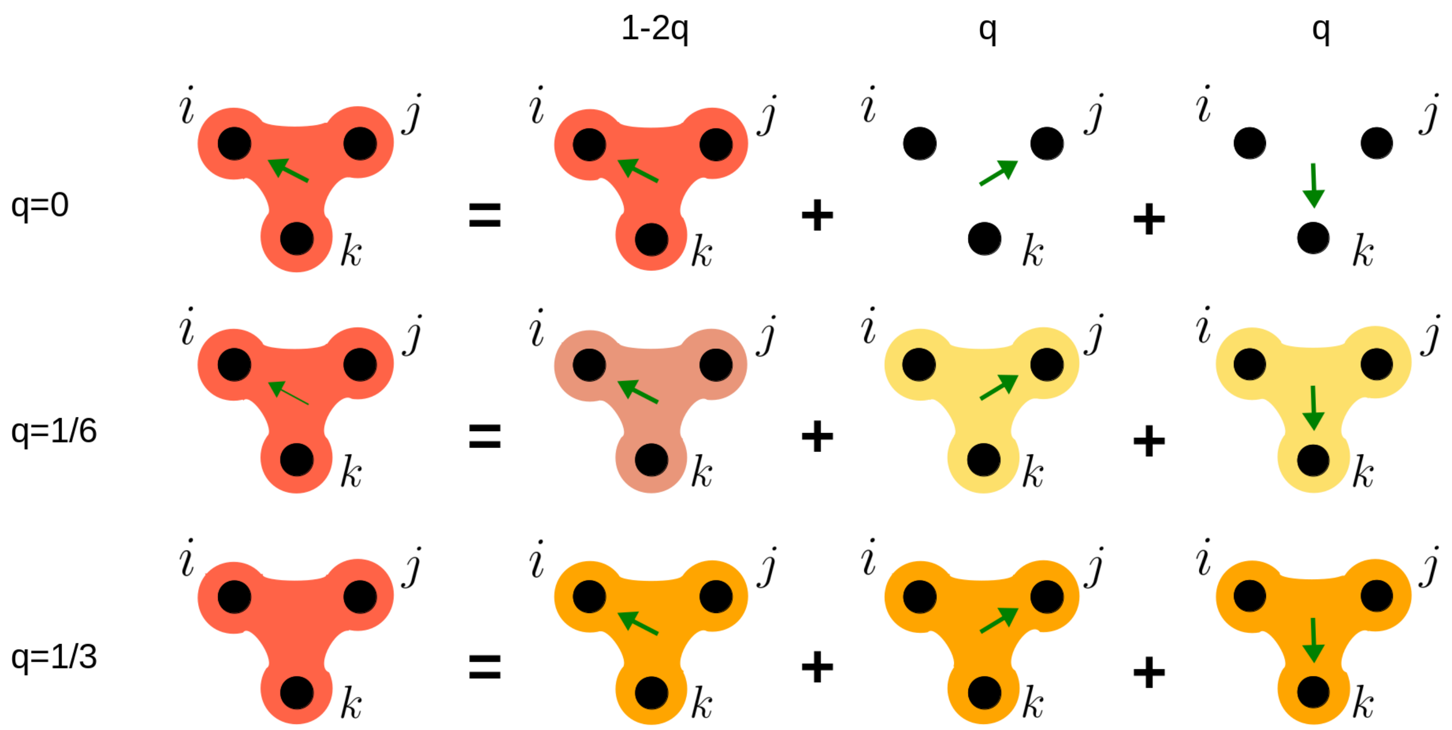



each component of the  tensor is symmetric

when p=1 the tensor is symmetric



# Alternative symmetrization




the total coupling strength is conserved

# Alternative symmetrization