Contagion-mitigating control in dynamic financial networks

Anton V. Proskurnikov
Polytechnic University of Turin, Italy

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Joint works with G. Calafiore and G. Fracastoro

- Control of Dynamic Financial Networks (The Extended Version), arXiv:2205.08879 (published in L-CSS)
- Clearing Payments in Dynamic Financial Networks, arXiv:2201.12898 (accepted by Automatica)
General motivation. Systemic risk in financial networks

- Highly interconnected structure.
- Complex structure of mutual obligations, shares in common assets etc.
- Many channels of financial “contagion”.

- Single fault can threaten the stability of the entire financial system

Schweitzer et al., Economic Networks: The New Challenges, Science, 325(5939)


Systemic risk theory:

How stresses, such as bankrupts and failures, in one part of a financial system can spread to its other parts?
Structure of the talks: 3 topics

- Multi-stage dynamic generalization of the E-N model: clearing as optimal control
- Contagion-mitigating interactions
A simple numerical example: Propagation of liquidity shortages (I)

System of banks A,B,C,D

- **Outside assets** (arcs coming from outside), e.g. bank A is owed 120 by households and companies.

- **Arcs** = payment obligations of one bank to another bank (some liquidity provided via loans etc.)

- Obligation to the **external sector** (households, companies etc.)

- The difference between assets and liabilities is a **net worth** (the net worths of banks A,B,C are 10, the net worth of D is 4).

- **Net worth cannot be negative**

Glasserman and Young, Contagion in Financial Networks, Journal of Economic Literature 54(3)
Occasionally, the assets of some bank drop, e.g. households delay or stop their payments. Then the total asset becomes less than the debt to pay.

Bank has to reduce all payments to the creditors. Usually, proportionality (pro rata) rule is applied: C pays to A,D, external sector in the proportion 2:2:1.

Assumption: external payments can be reduced.

The defaulting bank has to pay the maximal possible amount (priority rule), it cannot accumulate its net worth.
A simple numerical example: How defaults propagate (III)

- Hence, problems of a single bank cause problems to its creditors whose assets **have decreased**.

- These banks, in turn, have to reduce their payments to their creditors and to the outside sector, which may cause an «avalanche» of defaults.
A simple numerical example: How defaults propagate (IV)

- This chain of defaults can be cyclic and return to the «problematic» bank.

- We assume that C has 110 as asset, whereas in reality B has to decrease its payment! So C will not get more than 102! We need to recalculate its payments.

- What is the fair structure of payments in the situation where one or several banks default due to the outside assets’ drop? How to minimize the total loss?

- In reality, if you continue the iterative procedure, it will converge to a fixed point.
Iterative ‘fictitious default’ algorithm – converges to an equilibrium

• The procedure of iterative payment reduction (previous slides) gives an idea on how one channel of contagion (liquidity shortage) works. Can be turned into a model of defaults propagation, adding technical details (not the goal of E-N work!)

• Can be written as an iteration of a monotone operator on a lattice, a fixed point always exists due to Knaster-Tarski theorem (and, generically, is unique)

• The focus in E-N network is on the equilibria states: The actual payments after the renegotiations. Needed to evaluate the consequences and total loss of contagion. Have to be efficiently computable.

• Clearing in systemic risk = total debt reduction via mutual reimbursements.
Clearing: a toy example

- Two financial agents A and B
- A lends 1 bln $ to B (directed arc = obligation)
- For some reasons, A needs money and borrows 900 mln $ from B in credit
- If B fails to return its credit due to some problems, then A seeming suffers huge loss...
- But: if A pays its own debt (0.9), then B can repay it back, and the loss of A is 10 times smaller.
- Mutual reimbursements reduce the total loss in the system: the idea of clearing

Y.M. Kabanov et al., Clearing in financial networks//Theory Probab. Appl., 2018, 62(2)
Eisenberg-Noe (2001) model. (i) Obligations

- Directed graph of obligations
- Weights on arcs = debts to be paid

- Nominal outside assets = money to be raised from the non-financial sector

- Net worth of node is the difference of
  - the in-flow
  - and the outflow

\[ \bar{w}_i = \bar{c}_i + \sum_{k \neq i} \bar{p}_{ki} - \sum_{k \neq i} \bar{p}_{ki} \]

- Normally, the net worth is non-negative

- Abnormal case: the external assets are lost or less than their nominal values. Defaults of some banks propagate through the network.

- How much should defaulting banks actually pay to the others?
Eisenberg-Noe model of a static clearing procedure. (ii) Actual payments

**Rule 1: Limited liability.** Payments do not exceed their nominal values, the net value of each bank remains nonnegative.

\[ 0 \leq p_{ij} \leq \tilde{p}_{ij}, \quad p_i = \sum_{j \neq i} p_{ij} \leq c_i + \sum_{k \neq i} p_{ki} \]

**Rule 2: Absolute debt priority.**
Bank pays out its full debt \( \tilde{p}_i \) or its balance.

\[ p_i = \min \left( \tilde{p}_i, c_i + \sum_{k \neq i} p_{ki} \right) \quad (+) \]

\[ \tilde{p}_i = \sum_{k \neq i} \tilde{p}_{ik} \]

**Rule 3: Proportionality (pro-rata) rule.**
The larger debt, the larger actual payment is

\[ p_{ij} \sim \tilde{p}_{ij} \iff p_{ij} = a_{ij}p_i, \quad a_{ij} = \frac{\tilde{p}_{ij}}{\tilde{p}_i} \quad (*) \]

- Guarantees each bank a proportion in its debtors’ assets.
- Reduces the number of unknowns: matrix determined by \( p \)
A bunch of problems addressed in financial mathematics literature:

- Does the solution (clearing vector) exist?
- Is it unique?
- How to find it?

- What if we discard the pro-rata rule or replace it by an alternative division rule? Will this mitigate the loss caused by the defaults?
Existence of the **clearing vector**: nonlinear equations are solvable

\[ p_i = \min(\bar{p}_i, c_i + \sum_{k \neq i} a_{ki} p_k) \forall i \iff p = T_c(p) \]

\( p \) **is called a clearing vector.**

- The set of clearing vectors is always non-empty
- The maximal (or dominant) clearing vector \( p^* \) exists such that dominates any other clearing vector \( p \), that is, \( p \leq p^* \).
- Furthermore, \( p^* \) is the maximal (with respect to \( \leq \)) element of the polytope
  \[ \mathcal{D} = \{ p \in \mathbb{R}^N : 0 \leq p \leq \bar{p}, c + A^T p \geq p \} \neq \emptyset. \]
- In particular, \( p^* \) delivers optimum in the LP problem
  \[ L(p) = \sum_{i,j} (\bar{p}_{ij} - p_{ij}) = \sum_i (\bar{p}_i - p_i) \to \min \quad \text{subject to } p \in \mathcal{D}. \]
  The **“loss function”** \( L(p) \) can be replaced by any strict decreasing function.

- Alternative way: the fixed-point iteration (“fictitious default algorithm”)
  \[ p(n + 1) = T_c(p(n)), \quad p(0) = \bar{p} \]

Uniqueness of the clearing vector: sufficient conditions

\[ p_i = \min(\bar{p}_i, c_i + \sum_{k \neq i} a_{ki} p_k) \forall i \iff p = p^*. \]

• The clearing vector may be non-unique. Such situations are, however, non-generic.

• The clearing vector is unique if and only if every non-trivial (>1 node) strongly connected sink component either contains a node with \( c_i > 0 \)
  or
  is reachable from one of such nodes. Some graph theory and matrix analysis are needed.

• Otherwise, it is possible to describe the set of all clearing vectors

• More general cases (negative vector \( c \) or replacement by pro rata by other division rules): Massai, Como, Fagnani (2022); Herings and P. Csoka (2021)
The price of pro-rata rule.

• Pro-rata rule is often considered as natural for a number of reasons and is included in the bankruptcy law in many countries. Signing a contract, we want to guarantee some portion of the debtor’s assets in case of its fault.

• The pro-rata rule visible reduces the set of clearing matrices.

• It can be expected that the overall loss can be substantially decreased by removing it.

• We can find the clearing matrix delivering minimum to the overall loss (also reduces to LP, see details in the proceedings). Optimal matrix is generally non-unique.

\[ L(p) = \sum_{i,j} (\bar{p}_{ij} - p_{ij}) = \sum_i (\bar{p}_i - p_i) \rightarrow \text{min}. \]

The price of pro-rata rule: numerical tests

• The standard Erdős–Rényi random graph $G(n,p)$ is considered, $n=50$, average node degree varies from 0 to $np=35$. One fictitious sink node is added.

• Nominal liability of each arc chosen at random from $[0,100]$ (uniform distribution).

• Vector $c$ designed to provide positive yet small net worths (see details in the proceedings).

• Shocks are applied: outside asset of one randomly chosen bank is nullified.

• The following ratio may be considered as the “gain” of the pro-rata rule discarding:

$$G = \frac{L_{\text{min,pro-rata}} - L_{\text{min}}}{L_{\text{min,pro-rata}}} = 1 - \frac{L_{\text{min}}}{L_{\text{min,pro-rata}}}$$

• Another measure is the number of defaulting banks under the optimal choice of payments. Bank defaults if its actual payment is less than the nominal one: $p_i < \bar{p}_i$
The price of pro-rata rule: fair locally, non-optimal globally

- Visible decrease of overall loss as the graph becomes denser (the average degree $np$ grows). The number of defaults without pro-rata rule is also less than with pro-rata rule.
Pro-rata vs. free payment matrix (isolation of a problematic node)

(a) pro-rata payments

(b) full-matrix payments

All banks in default, total unpaid amount is 47.28

1 bank in default, total unpaid amount is 20


0 – fictitious node, corresponds to the payments to non-financial sector
Structure of the talks: 3 topics

• Eisenberg-Noe model: contagion via propagation of liquidity shortage. Clearing payments.

• Multi-stage dynamic generalization of the E-N model: clearing as optimal control

• Contagion-mitigating interactions
Principal limitation of the E-N model

The EN model describes static clearing process
• all outside assets are available at once;
• all liabilities are claimed and due simultaneously;
• all clearing payments are computed simultaneously;

Next step: non-static (multi-step) clearing models:
• financial operations are allowed for a given number of time periods after the initial shocks;
• some nodes may actually recover and eventually manage to fulfill their obligations if the liquidity inflow continues;
• we do not freeze operations in case of instantaneous default, but allow for a grace period and carry over the residual liabilities for the next time slot
Model 1: Optimal multi-period clearing procedure

operations over $T$ consecutive periods

\[
L = \sum_{t=0}^{T-1} \sum_{i,j} (\bar{p}_{ij}(t) - p_{ij}(t)) \rightarrow \min
\]

at each period $t=0,1,..., T-1$, bank $i$ is characterized by:

- outside asset: external input
- nominal liabilities to other banks: dynamic variable
- net worth in the previous periods: dynamic variable

\[
c_i(t) \geq 0 \\
\bar{p}_{ij}(t) \geq 0 \\
w_i(t)
\]

The actual payments are also distributed over $T$ periods

\[
p_{ij}(t)
\]

- Limited liability rule (pay no more than liability, the balance remains non-negative)
- We do not impose absolute debt priority (non-convex!)
- The residual liabilities are transferred to future periods, and the interest rate can apply
- Pro-rata rule with the proportions are determined at the initial step

\[
0 \leq p_{ij}(t) \leq \bar{p}_{ij}(t) \\
w_i(t + 1) = w_i(t) + c_i(t) + \sum_{k \neq i} p_{ki}(t) - \sum_{k \neq i} p_{ik}(t) \geq 0
\]

\[
\bar{p}_{ij}(t + 1) = \alpha(\bar{p}_{ij}(t) - p_{ij}(t))
\]

\[
p_{ij}(t) = a_{ij}p_i(t) \\
a_{ij} = \frac{\bar{p}_{ij}}{\bar{p}_i} \\
p_i(t) = \sum_{j \neq i} p_{ij}(t)
\]
Optimal multi-period clearing procedure: equivalent LP

\[ L = \sum_{t=0}^{T-1} \sum_{i,j} (\bar{p}_{ij}(t) - p_{ij}(t)) = a_0 \mathbf{1}^\top \bar{p} - \sum_{t=0}^{T-1} a_t \mathbf{1}^\top p(t) \rightarrow \min_{p(0), \ldots, p(T-1)} \]

\[ a_t = \sum_{j=0}^{T-t-1} \alpha^j = \begin{cases} \frac{\alpha^{T-t-1}}{\alpha-1}, & \text{if } \alpha > 1 \\ T-t, & \text{if } \alpha = 1 \end{cases} \quad (a_0 > a_1 > \ldots > a_{T-1}) \]

\[ \text{subject to} \]

\[ p(t) \geq 0, \]

\[ \sum_{k=0}^{t} \alpha^{t-k} p(k) \leq \alpha^t \bar{p} \]

\[ \sum_{k=0}^{t} c(k) + \sum_{k=0}^{t} (A^\top p(k) - p(k)) \geq 0 \]

\[ \forall t = 0, 1, \ldots, T-1. \]
Optimal solution – the counterpart of the maximal clearing vector

- Optimal solution exists and is unique for each sequence of outside assets \( c(0), \ldots, c(T-1) \): (a non-trivial property: solution to LP may be non-unique!)

- **Absolute debt priority is implied by the optimality:**
  \[
  p_i^*(t) = \min \left( \bar{p}_i(t), w_i(t) + c_i(t) + \sum_{k \neq i} p_{ki}^*(t) \right), \quad p_{ki}^*(t) = a_{ki}p_k^*(t).
  \]

- **Causality:** in fact, the optimal value \( p^*(t) \) depends on \( c(0), \ldots, c(t) \)

- **Greedy strategy is optimal:** in fact, \( p^*(t) \) minimizes the loss in the static E.-N. problem

\[
L_t(p) = \sum_i (\bar{p}_i(t) - p_i) \rightarrow \min_p \quad \text{subject to} \quad 0 \leq p \leq \bar{p}(t), \ w(t) + c(t) + A^\top p \geq p.
\]

- Instead of solving LP with \( Tn \) scalar variables and \( 3Tn \) constraints, one can solve a sequence of \( T \) LP with \( n \) variables and \( 3n \) constraints:

\[
[p = \bar{p}(0), w(0) = 0] \xrightarrow{c(0)} p^*(0) \xrightarrow{c(1)} [p(1), w(1)] \xrightarrow{c(2)} p^*(1) \xrightarrow{} [p(2), w(2)] \xrightarrow{} \ldots
\]
Model 2: Same as Model 1, but without pro-rata rule

\[ L = \sum_{t=0}^{T-1} \sum_{i,j} (\bar{p}_{ij}(t) - p_{ij}(t)) = a_0 \mathbf{1}^\top \bar{P} \mathbf{1} - \sum_{t=0}^{T-1} a_t \mathbf{1}^\top P(t) \mathbf{1} \rightarrow \min_{P(0), \ldots, P(T-1)} \]

subject to

\[ P(t) \geq 0, \]
\[ \sum_{k=0}^{t} \alpha^{t-k} P(k) \leq \alpha^t \bar{P} \]
\[ \sum_{k=0}^{t} c(k) + \sum_{k=0}^{t} (P(k)^\top \mathbf{1} - P(k) \mathbf{1}) \geq 0 \]
\[ w_i(t + 1) \geq 0 \forall i \]
\[ \forall t = 0, 1, \ldots, T - 1. \]
Properties of the optimal solution without the pro-rata rule

• Optimal solution exists but is non-unique even for $T=1$

• No causality: in fact, the optimal matrices $P^*(t)$ depend on all $c(0),...,c(T-1)$

• **Absolute debt priority:** $p^*_i(t) = \min\left(\bar{p}_i(t), w_i(t) + c_i(t) + \sum_{k \neq i} p_{ki}(t)\right)$

• **Greedy strategy is generally sub-optimal**, the LP **cannot be solved** as a sequence of $T$ smaller problems

• The key advantage: the total loss and number of defaulting banks reduces
Structure of the talks: 3 topics

- Multi-stage dynamic generalization of the E-N model: clearing as optimal control
- Contagion-mitigating interactions
Some authorities (central banks etc.) can control the financial network by optimal injections of cash at nodes.

\[ [u] = (u(0), \ldots, u(T - 1)) \]

\[ c(t) = e(t) + u(t) \]
Cost function: same as before or even slightly more general

\[ J([p], [u]) = (1 - \eta)L([p]) + \eta 1^\top \bar{p}(T) + \gamma B(T - 1) \]

- We penalize the total loss (same as before)
- + terminal cost (zero if no bank is at default)
- + the total budget used to help the banks

\[ B(s) = \sum_{t=0}^{s} \sum_{i=1}^{n} u_i(t) \]
We can define the optimal control problem as an LP problem, imposing

- limited liability (rewritten): 
  \[ p(t) \geq 0, \]
  \[ \sum_{k=0}^{t} \alpha^{t-k} p(k) \leq \alpha^{t} \bar{p} \]
  \[ \sum_{k=0}^{t} c(k) + \sum_{k=0}^{t} \left( A^\top p(k) - p(k) \right) \geq 0 \]
  \[ \forall t = 0, 1, \ldots, T - 1. \]

- limited budget for controlling the network: 
  \[ B(t) \leq F(t) \]
Numerical example: injected liquidity vs. potential loss

- Without interventions **all nodes default**, total loss is **49.92**.
- We consider the control problem over $T=3$ periods with $F(0)=15$, $F(1)=30$, $F(2)=50$, the optimal control is
  
  \[
  u(0) = \begin{bmatrix} 2.19 \\ 5.25 \\ 0 \\ 0 \\ 5.20 \\ 2.36 \\ 0 \end{bmatrix}, \quad u(1) = \begin{bmatrix} 2.84 \\ 0 \\ 0 \\ 1.9 \\ 0 \\ 0 \end{bmatrix}, \quad u(2) = 0
  \]

- **No bank is at default** at $T=3$
  - total injected liquidity is **19.74**.

\[e(0) = (105, 25, 10, 190, 10, 120, 0)\]
Properties of optimal solutions

\[
J([p], [u]) = (1 - \eta)L([p]) + \eta \mathbf{1}^\top \bar{p}(T) + \gamma B(T - 1)
\]
\[
\eta \in [0, 1), \gamma > 0
\]

- **The debt priority rule is respected:**
  \[
  p^*_i(t) = \min \left( \bar{p}_i(t), w_i(t) + e_i(t) + u^*_i(t) + \sum_{k \neq i} p^*_{ki}(t) \right),
  p^*_{ki}(t) = a_{ki}p^*_k(t).
  \]

- **The bank utilizes liquidity immediately by paying out all its balance**
  \[
  u^*_i(t) > 0 \implies w_i(t + 1) = 0,
  \]

- **Additional liquidity is provided as early as possible to each bank:**
  \[
  u^*_i(t_0) = 0, B^*(t_0) < F(t_0) \implies u^*_i(t) = 0 \forall t \geq t_*.\]
Conclusion

• We propose a novel dynamic model of clearing in financial networks.

• The model departs from the classical Eisenberg-Noe model and inherits some its properties, e.g., uniqueness of the optimal solution under the pro-rata rule.

• Relaxing the pro-rata constraint, one can substantially decrease the number of defaulting banks and the total loss, however, the problem becomes more complicated and cannot be solved stepwise.

• We consider optimal control interventions aimed at mitigating the damage of an initial financial shock

• Further extensions: robust control in the face of uncertainties in outside assets and nominal debts. Calafiore et al., "Control of Dynamic Financial Networks,” L-CSS, vol. 6, 2022

• Future works: more realistic models (common illiquid assets etc.), MPC-like control
Thank you!