# Energy-Aware Controllability of Complex Networks

Giacomo Baggio[1], Fabio Pasqualetti[2], Sandro Zampieri[1]

[1]University of Padova

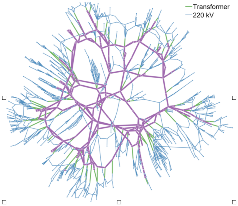
[2]University of California, Riverside

September 22, 2023

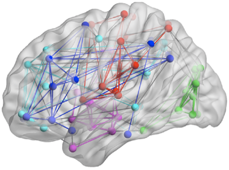
# Table of Contents

# Motivations: need of tools for dynamical network analysis

Social network




Power network



Brain network

- Static networks: connectivity, centrality, modularity, ...
- Dynamical networks: equilibria, stability, basins of attraction, ....

**Controllability** as a natural metrics of the interaction strength in dynamic networks

Questions:

- Is the network controllable?
- How many control nodes?
- How to select control nodes?

# Literature

Y. Y. Liu, J. J. Slotine, A. L. Barabasi, "Controllability of complex networks," Nature 2011.

# Literature

Y. Y. Liu, J. J. Slotine, A. L. Barabasi, “Controllability of complex networks,” Nature 2011.

## Network science


- J. Sun and A. E. Motter, “Controllability transition and nonlocality in network control,” Physical Review Letters, 2013.
- Liu, Yang-Yu, Jean-Jacques Slotine, and Albert-László Barabási. "Observability of complex systems." Proceedings of the National Academy of Sciences 110.7 (2013): 2460-2465.
- Nepusz, Tamás, and Tamás Vicsek. "Controlling edge dynamics in complex networks." Nature Physics 8.7 (2012): 568-573.
- Wang, W. X., Ni, X., Lai, Y. C., Grebogi, C. (2012). Optimizing controllability of complex networks by minimum structural perturbations. Physical Review E, 85(2), 026115.
- Yuan, Z., Zhao, C., Di, Z., Wang, W. X., , Lai, Y. C. (2013). Exact controllability of complex networks. Nature communications, 4.
- Mones, Enys, Lilla Vicsek, and Tamás Vicsek. "Hierarchy measure for complex networks." PloS one 7.3 (2012): e33799.

......

......

- Alizadeh, S., Pósfai, M., Ghasemi, A. (2023). Input node placement restricting the longest control chain in controllability of complex networks. Scientific Reports, 13(1), 3752.
- D’Souza, R. M., di Bernardo, M., Liu, Y. Y. (2023). Controlling complex networks with complex nodes. Nature Reviews Physics, 5(4), 250-262.

# Literature

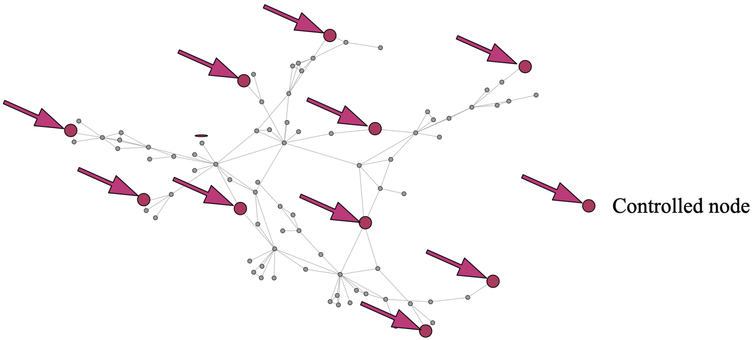
Y. Y. Liu, J. J. Slotine, A. L. Barabasi, "Controllability of complex networks," Nature 2011.

## Life science

- Branchi, I. (2022). Recentering neuroscience on behavior: the interface between brain and environment is a privileged level of control of neural activity. **Neuroscience , Biobehavioral Reviews**, 138, 104678.
- Esmaeilzadeh, A. A., Kashian, M., Salman, H. M., Alsaffar, M. F., Jaber, M. M., Soltani, S., ... , Kastelic, J. W. (2022). Identify Biomarkers and Design Effective Multi-Target Drugs in Ovarian Cancer: Hit Network-Target Sets Model Optimizing. **Biology**, 11(12), 1851.
- Craske, M. G., Herzallah, M. M., Nusslock, R., Patel, V. (2023). From neural circuits to communities: an integrative multidisciplinary roadmap for global mental health. **Nature Mental Health**, 1(1), 12-24.
- Vilonen, L. L., Hoosein, S., Smith, M. D., Trivedi, P. (2023). Legacy effects of intensified drought on the soil microbiome in a mesic grassland. **Ecosphere**, 14(6), e4545.
- Ginsberg, S. D., Sharma, S., Norton, L., Chiosis, G. (2023). Targeting stressor-induced dysfunctions in protein–protein interaction networks via epichaperomes. **Trends in Pharmacological Sciences**.
- Schmälzle, R., , Huskey, R. (2023). Integrating media content analysis, reception analysis, and media effects studies. **Frontiers in Neuroscience**, 17, 1155750.
- Zhang, X., Pan, C., Wei, X., Yu, M., Liu, S., An, J., ... , Zhang, W. (2023). Cancer-keeper genes as therapeutic targets. **Iscience**, 26(8).

# Table of Contents

# Controllability




$n$ states system with dynamics

$\dot{x}_i(t) = \sum_j A_{ij} x_j(t) + u_i(t)$ controlled node

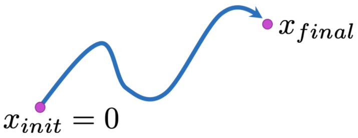
$\dot{x}_i(t) = \sum_j A_{ij} x_j(t)$ uncontrolled node

$$\dot{x}(t) = Ax(t) + Bu(t)$$

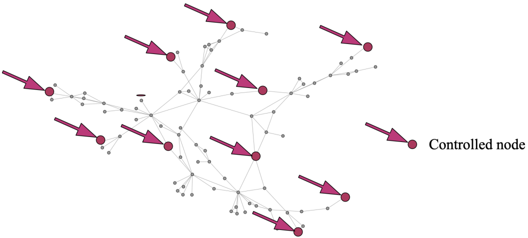
# Controllability

$$\dot{x}(t) = Ax(t) + Bu(t)$$

**Controllability**: We can reach any final state by acting on the input signal u(t)



# Popular approach: Structural Controllability



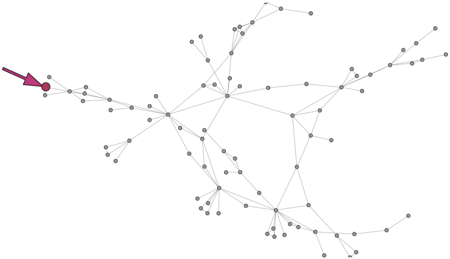

$$\dot{x}(t) = Ax(t) + Bu(t)$$

Only zero/nonzero pattern of $A$ is imposed.

**Structural controllability**: Given a non-zero pattern of the matrix $A$, the system is structurally controllable, if it is controllable for (almost) every choice of the weights on the non-zero positions.

- **Pros:** It allows graph theoretic analysis.
- **Cons:** It hides a certain kind of uncontrollability when $n$ is large.

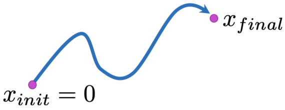
# Warnings on Structural Controllability



- If the graph is strongly connected network with self loops ($A_{ii} \neq 0$) then the the system is structurally controllable from any **single node**.
- However, this seems to be in some sense unrealistic in practice.
- Need to introduce an **energy aware** notion of controllability.

Yan, G., Ren, J., Lai, Y. C., Lai, C. H., Li, B. Controlling complex networks: How much energy is needed?. Physical review letters, 108(21), 2012.
F Pasqualetti, S Zampieri, F Bullo, Controllability metrics, limitations and algorithms for complex networks, IEEE Transactions on Control of Network Systems 1 (1), 40-52, 2014

# Energy related controllability metrics




$$Energy(u(t)) := \int_0^\infty \|u(t)\|^2 dt$$

$$Opt - Energy(u(t)) = x_{final}^T W^{-1} x_{final}$$

where $W$ is the controllability Gramian

$$W := \int_0^\infty e^{At} B B^T e^{A^T t} dt$$

High controllability $\Leftrightarrow$ **Large Gramian** $\Leftrightarrow$ Small control energy

Scalar metrics: $\lambda_{\min}(W), \lambda_{\max}(W), \frac{1}{n} tr(W), \frac{1}{n} tr(W^{-1}), \det(W)^{\frac{1}{n}}$

# Table of Contents

# Energy related controllability metrics

$$Opt - Energy(u(t)) = x_{final}^T W^{-1} x_{final}$$

- $\lambda_{\max}(W), tr(W)$ are more mathematical treatable but with a less control meaning.
- $\lambda_{\min}(W)^{-1}$ is the worst case control energy for unit norm target states.
- $\frac{1}{n} tr(W^{-1})$ is the average control energy for unit norm target states.
- $\det(W)$ is the volume of the state space that is reachable by norm one inputs.

# Energy related controllability metrics

$$\lambda_{\min}(W) \leq \frac{n}{tr(W^{-1})} \leq \det(W)^{1/n} \leq \frac{tr(W)}{n} \leq \lambda_{\max}(W)$$

- $\lambda_{\min}(W)$ and $\frac{n}{tr(W^{-1})}$ are equivalent since

$$\frac{\lambda_{\min}(W)}{n} \leq \frac{n}{tr(W^{-1})} \leq \lambda_{\min}(W)$$

- $\lambda_{\max}(W)$ and $\frac{tr(W)}{n}$ are equivalent since

$$\frac{\lambda_{\max}(W)}{n} \leq \frac{tr(W)}{n} \leq \lambda_{\max}(W)$$

In the sequel we analize $\lambda_{\min}(W)$ and $\lambda_{\max}(W)$.

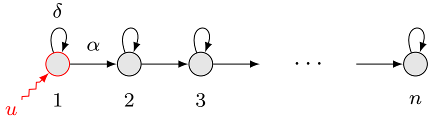
# Energy related controllability metrics

- **Pros:** Good metrics of controllability for large scale dynamical networks
- **Cons:** Difficult to relate to the network $A$ and to the control nodes allocation $B$

Need to find good **proxies** of the controllability degree:
Controllability increases when

1. number $m$ of input increases
2. when $A$ approaches instability
3. when $A$ displays a "spatial" instability

# Example: Line



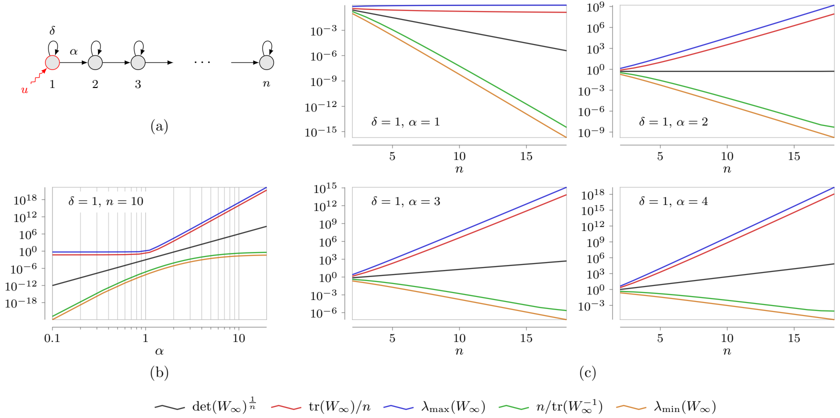

$$\dot{x}_i(t) = -\delta x_i(t) + \alpha x_{i-1}(t)$$

where $\alpha, \delta > 0$.

$$A = \begin{bmatrix} -\delta & 0 & 0 & \cdots & 0 \\ \alpha & -\delta & 0 & \cdots & 0 \\ 0 & \alpha & -\delta & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \alpha & -\delta \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{n\times 1},$$

As expected all controllability indices grow as $\delta \to 0$

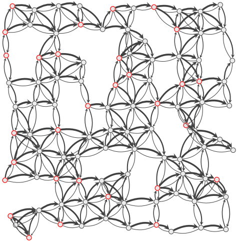
# Example: Line



# Example: Line

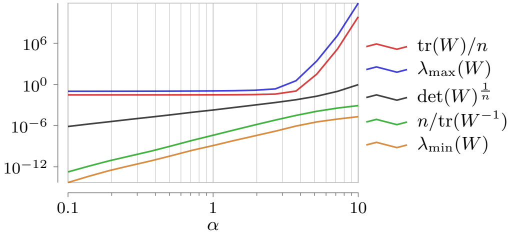
| metric | asymptotic behavior |
|---|---|
| $\lambda_{\min}(W)$<br>$n/\mathrm{tr}(W^{-1})$ | $\left(\frac{\alpha}{\alpha+2\delta}\right)^{2n}$ |
| $\det(W)^{1/n}$ | $\left(\frac{\alpha}{2\delta}\right)^{2n}$ |
| $\lambda_{\max}(W)$<br>$\mathrm{tr}(W)/n$ | $\left(\frac{\alpha}{\delta}\right)^{2n}$ if $\alpha > \delta$<br>decays polynomially in $n$ if $\alpha < \delta$ |

# Example: Random geometric



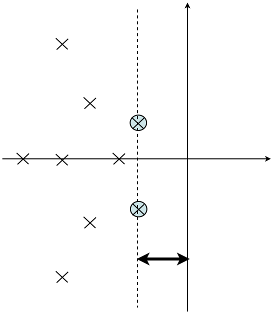
$n = 100,\ m = 30,\ \delta = 5$

(a)




(b)

# Controllability estimates



Controllability is influenced by

- Number $m$ of inputs
- Distance to instability of $A$=spectral abscissa of $A$

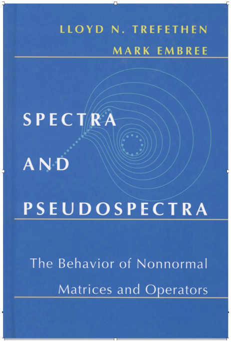
$$\rho(A) := \max\{Re[\lambda] : \lambda \text{ eigenvalues of } A\} < 0$$

- Spatial instability=Degree of non-normality of $A$.

# Table of Contents

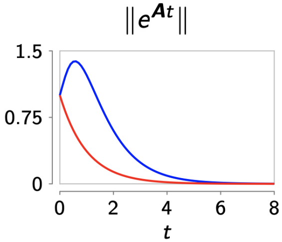
# Matrix non-normality

$A$ is normal if $AA^T = A^T A$. Otherwise it is non-normal.

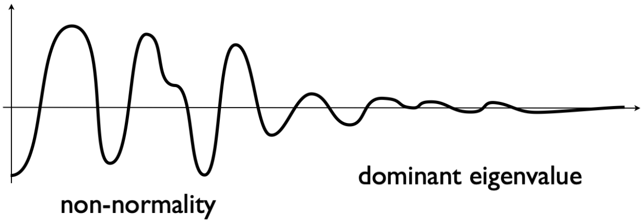


# Matrix non-normality

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$




$$A = \begin{bmatrix} -1 & 0 \\ 5 & -2 \end{bmatrix}$$



# Characteristics of normal matrices

For normal matrices we have:

1. $A$ is diagonalizable with eigenvector matrix $V$ having condition number $k(V) = 1$.
2. Let $\rho(A)$ is the spectral abscissa and $\omega(A) := \rho\left(\frac{A+A^T}{2}\right)$ (called the numerical abscissa). Then $\omega(A) = \rho(A)$.
3. The Shur form of $A$ is diagonal.

$$U^T A U = \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \lambda_n \end{bmatrix}$$

with $U$ orthogonal.

# Non-normality indices

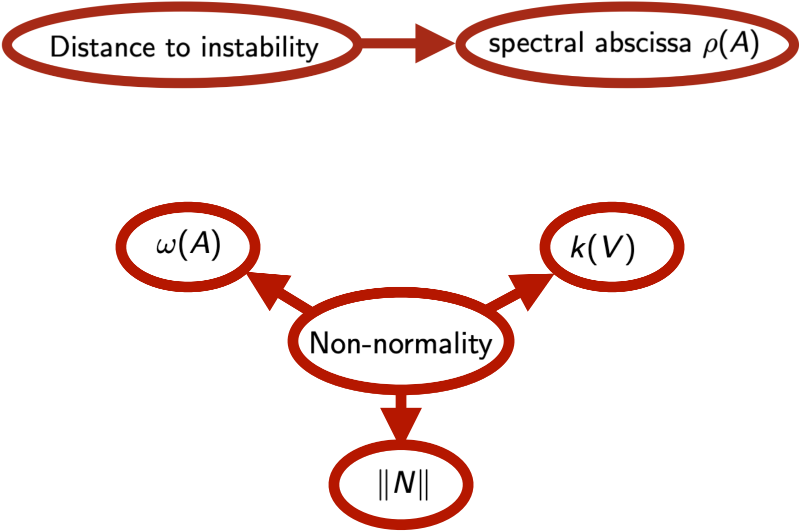
We can propose 3 ways to measure the matrix non-normality:

1. For diagonalizable non-normal matrices take the condition number $k(V)$ of the eigenvector matrix $V$ of $A$.
2. Take the gap between $\omega(A)$ and $\rho(A)$
3. Take the Shur form of $A$ and let $N$ be its strictly lower triangular part. We can take $\|N\|$ as a non-normality index

$$U^T A U = \Lambda + N$$

$$N = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ N_{2,1} & 0 & 0 & \cdots & 0 \\ N_{3,1} & N_{3,2} & 0 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ N_{n,1} & \cdots & N_{n,n-2} & N_{n,n-1} & 0 \end{bmatrix}$$

# Non-normality indices



# Table of Contents

# Upper bounds on $\lambda_{\max}(W)$

$$\lambda_{\max}(W) \leq \begin{cases} -\dfrac{1}{2\omega(A)} & \text{if } \omega(A) < 0 \\[2ex] cost\ \mu^n & \text{if } \omega(A) > 0 \end{cases}$$

where the exponent is

$$\mu = \left(\frac{\|N\|}{-\rho(A)}\right)^2$$

and $\mu > 1$ when $\omega(A) > 0$.

# Upper bounds on $\lambda_{\min}(W)$

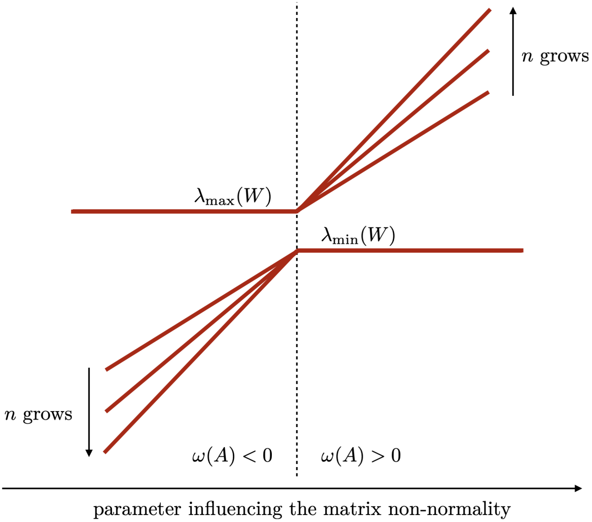
$$\lambda_{\min}(W) \leq \begin{cases} cost\ \nu^{\frac{n}{m}} & \text{if } \omega(A) < 0 \\ -\dfrac{m}{2tr(A)} & \text{if } \omega(A) > 0 \end{cases}$$

where the exponent is

$$\nu = \frac{\|A\| + \omega(A)}{\|A\| - \omega(A)}$$

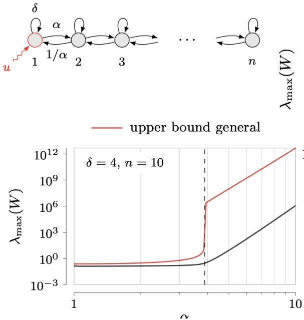
and $\nu < 1$ when $\omega(A) < 0$.

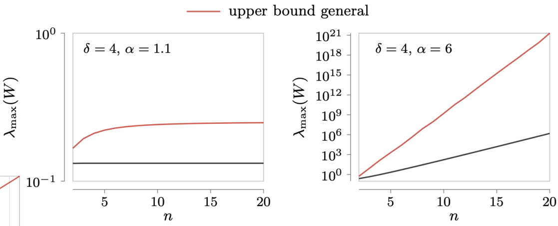
# Conceptual picture



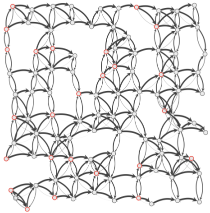
# Table of Contents

# Example: line

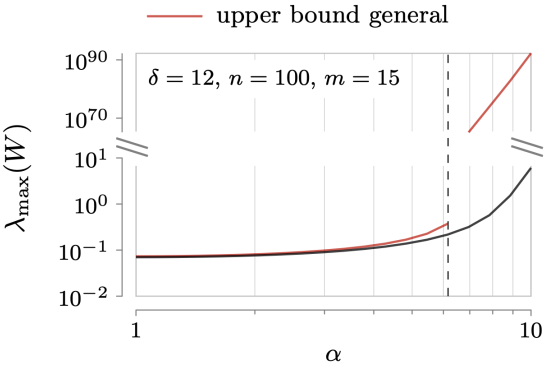


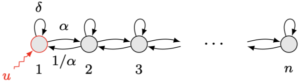
# Example: random geometric graph

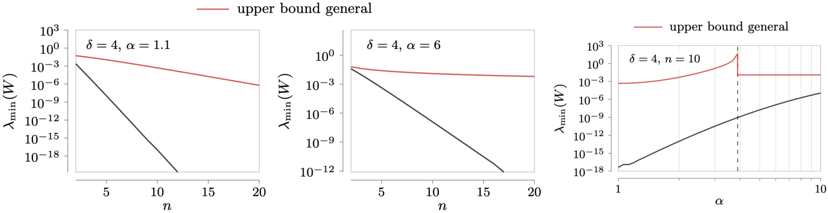


$n = 100,\ m = 25,\ \delta = 7$

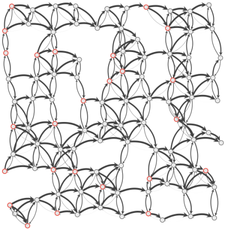


# Example: line

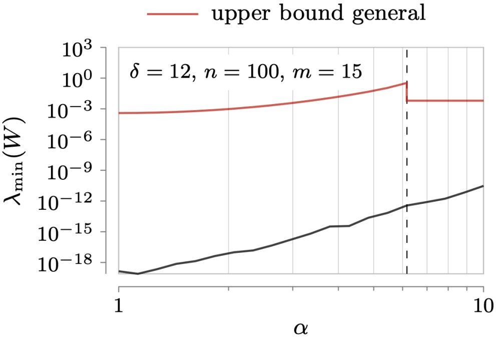


# Example: random geometric graph



$n = 100$, $m = 25$, $\delta = 7$



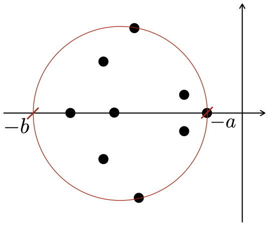
# Other upper bounds

Upper bounds involving $k(V)$ non-normality index

$$\lambda_{\max}(W) \leq \frac{k(V)^2}{-2\rho(A)} \qquad\qquad \lambda_{\min}(W) \leq \frac{k(V)^4}{2a}\nu^{\frac{n}{m}-1}$$

where the exponent is

$$\nu = \left(\frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{a}}\right)^2$$



# Table of Contents

# Concluding remarks

1. Take home message: Controllability improves when we add control nodes, we get close to instability, we have a **non-normal=anisotropic** network.
2. A general network wise characterization of non-normality is quite elusive.
3. Finding lower bounds is much harder because it involves the choice of a strategy for input nodes positioning.
4. For the single input case we have nice estimates of the exponential decay rate for the minimum eigenvalue and the determinant indices.
5. **More can be said for cycle free (tree) networks**.