Curtin University

FACULTY OF SCIENCE AND ENGINEERING

Adaptive-gain control for population dynamics: epidemic networks and evolutionary games

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- Introduction
  - Population dynamics
  - Background on SIS network model

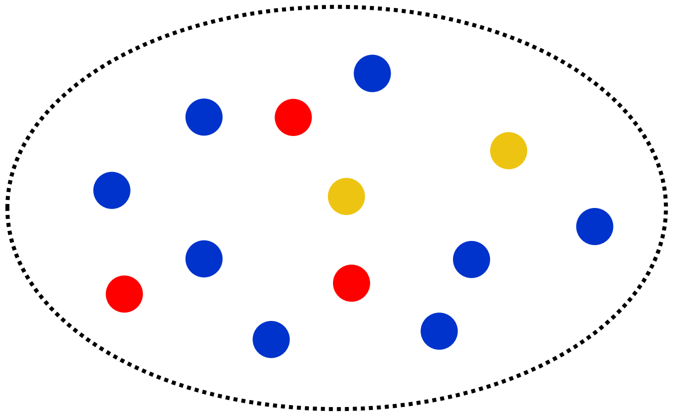
- Controlling epidemic spreading on a network
  - Full network control
  - Partial network control

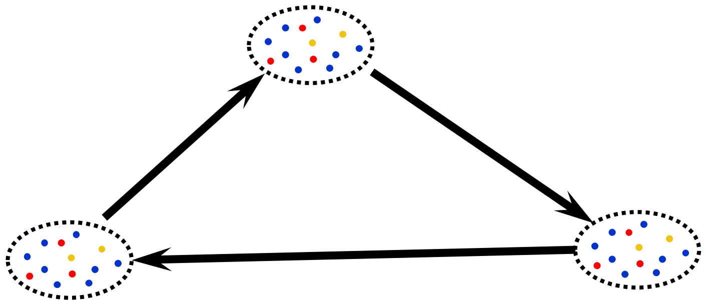
- Conclusions and future work

Population dynamics

A population of agents who repeatedly interact with one another, and consequently, agent states evolve over time

- Epidemic spreading: people interact physically to transmit infectious diseases [1]
- Evolutionary games: individuals engage in strategic interactions (games), e.g. genetic trait selection in evolutionary biology, or social dilemmas in human societies [2]
- Networks of populations (meta-populations) can be considered





[1] L. Zino and M. Cao. Analysis, prediction, and control of epidemics: A survey from scalar to dynamic network models. *IEEE Circuits and Systems Magazine*, 2021.

[2] W. H. Sandholm, *Population Games and Evolutionary Dynamics*. Cambridge University Press, 2010.

A population of agents who repeatedly interact with one another, and consequently, agent states evolve over time

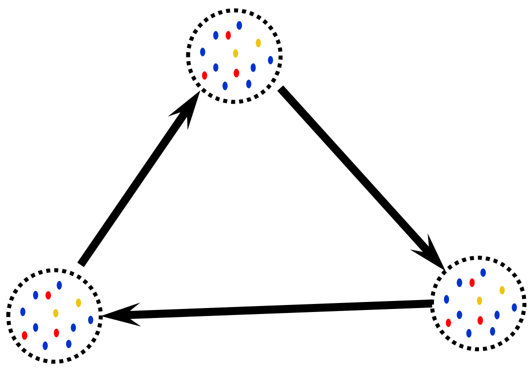
$$\dot{x}(t) = f(x(t)), \qquad x \in \mathbb{R}_+^n$$

Typical problems of interest

- Characterising the dynamics of the system as a function of system parameters (epidemic transmission rates, payoff functions of a game)
  - Number of equilibria (including when equilibria exist)
  - Stability of equilibria, including both local and global stability
  - Asymptotic behaviour: convergence to equilibrium, limit cycle, chaos, etc.

- Controlling the system and steering $x(t)$ to a desired point
  - Driving an epidemic model to the disease-free equilibrium
  - Steering a population game to a specific equilibrium, that might represent a consensus adoption of one strategy, e.g. one desirable genetic trait or social behaviour.

A motivating example: SIS Network Model



- Consider $n \geq 2$ large populations of constant size (birth rate = death rate)
- Each individual has two possible states: Susceptible (S) and Infected (I).
  Individuals recover with no immunity to the disease (e.g. influenza, gonorrhea [1])

Susceptible individuals

$$\dot{x}_i(t) = \underbrace{-d_i x_i(t)}_{\text{Recovery of infected individuals}} + \overbrace{(1 - x_i(t))}^{\text{Susceptible individuals}} \underbrace{\sum_{j \in \mathcal{N}_i} b_{ij} x_j(t)}_{\text{Infected individuals}}, \quad i \in \{1, 2, \ldots, n\}$$

Recovery of
infected individuals

Infected individuals

- $x_i = [0,1]$ is the proportion of population i that is infected
- $d_i > 0$ is the recovery rate
- $b_{ij} \geq 0$ is infection rate from population j individuals to population i individual.

[1] A. Lajmanovich and J. A. Yorke. A Deterministic Model for Gonorrhea in a Nonhomogeneous Population. *Mathematical Biosciences*, **28**(3-4): pp 221-236, 1976

Convergence and equilibria properties

Assume the network is strongly connected (equivalently, B is a nonnegative irreducible matrix)

$$\dot{x}(t) = \Big[- D + (I_n - \mathrm{diag}(x(t))B \Big] x(t)$$

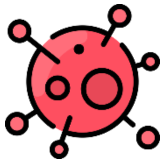
$$x = [x_1, x_2, \ldots, x_n]^\top \quad D = \mathrm{diag}(d_1, d_2, \ldots, d_n), \quad B = \{b_{ij}\}$$

Assume that $x(0) \in [0,1]^n$.
Then, $x(t) \in [0,1]^n$ for all $t \geq 0$.

- Define the reproduction number: $R_0 = \rho(D^{-1}B)$ where $\rho(D^{-1}B)$ denotes the spectral radius of $D^{-1}B$.

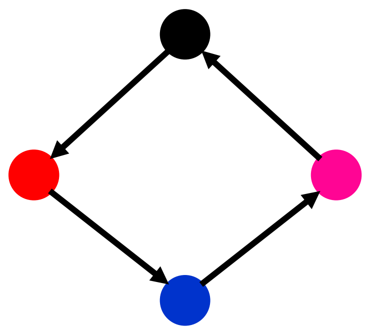
- $x = 0_n$ is the unique equilibrium if and only if $R_0 \leq 1$. Then, $x = 0_n$ is globally asymptotically stable (and exponentially stable if $R_0 < 1$)

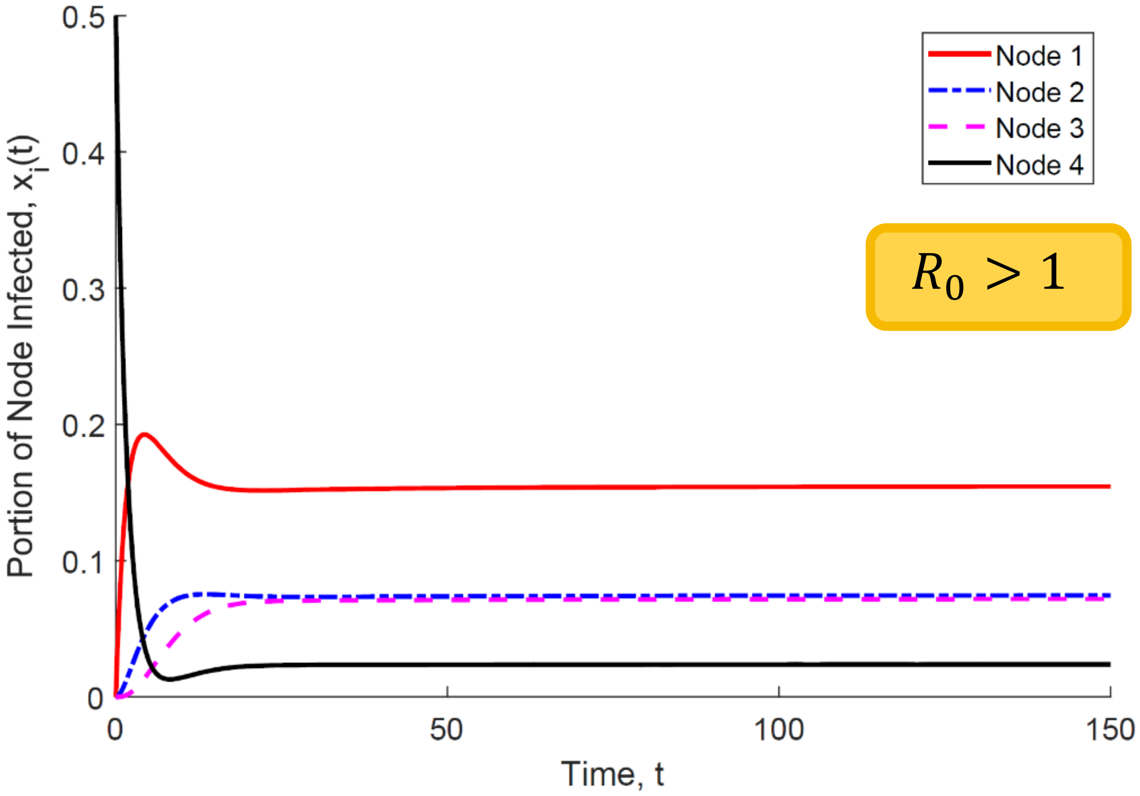
- If $R_0 > 1$, then 0_n is unstable. There exists a unique endemic equilibrium $x^* \in (0,1)^n$ that is exponentially stable for all $x(0) \neq 0_n$

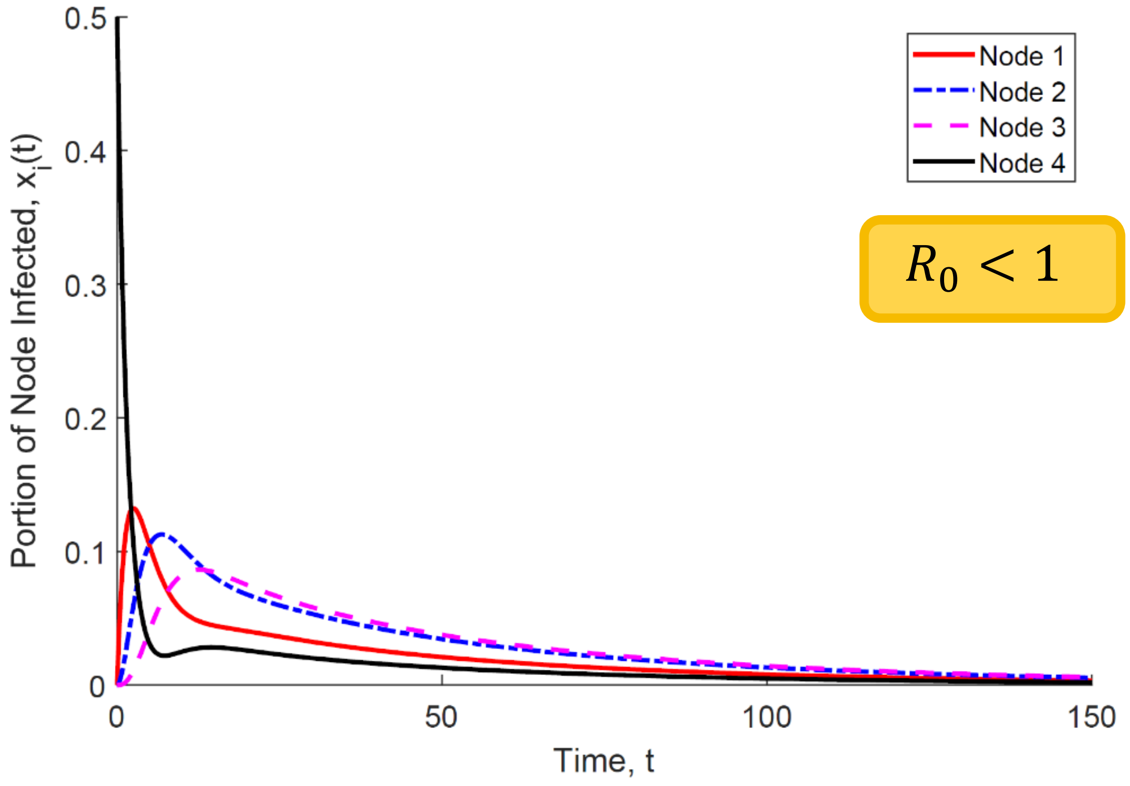


Simulation Example

Initial condition: $x(0) = [0, 0, 0, 0.5]^\top$



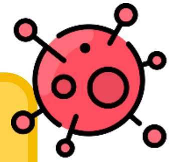




Control problem for SIS network model

$$\dot{x}(t) = \Big[-D + (I_n - \mathrm{diag}(x(t))B \Big] x(t)$$

$$x = [x_1, x_2, \ldots, x_n]^\top \qquad D = \mathrm{diag}(d_1, d_2, \ldots, d_n), \qquad B = \{b_{ij}\}$$



Basic problem: Assume that $R_0 > 1$. How can one adjust the values of d_i and b_{ij} so that, for all $x(0)$, we minimise $x_i(\infty)$ for all i (or some i)?

- Increasing recovery rate d_i → application of medical interventions, e.g. more doctors, medicines, anti-viral drugs.

- Decreasing infection rate b_{ij} → application of Nonpharmaceutical Interventions (NPIs), e.g. masks, mobility restrictions.

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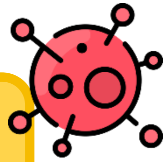

Basic problem: Assume that $R_0 > 1$. How can one adjust the values of d_i and b_{ij} so that, for all $x(0)$, we minimise $x_i(\infty)$ for all i (or some i)?

- Optimisation of network structure: minimise R_0 by
  - Removal of nodes/edges, or tuning of d_i or b_{ij}, against a fixed resource budget [1,2,3]
  - Requires global network information, and is a one-shot method

[1] V. L. Somers and I. R. Manchester. Sparse Resource Allocation for Control of Spreading Processes via Convex Optimization. *IEEE Control Systems Letters*, 2020

[2] P. Van Mieghem, D. Stevanovic, F. Kuipers, C. Li, R. Van De Bovenkamp, D. Liu, and H. Wang. Decreasing the spectral radius of a graph by link removals. *Physical Review E*, 2011.

[3] V. M. Preciado, M. Zargham, C. Enyioha, A. Jadbabaie, and G. Pappas. Optimal resource allocation for network protection: A geometric programming approach. *IEEE Transactions on Control of Network Systems*, 2014.

Control problem for SIS network model

$$\dot{x}(t) = \Big[- D + (I_n - \mathrm{diag}(x(t))B \Big] x(t)$$

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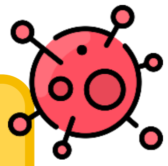


Basic problem: Assume that $R_0 > 1$. How can one adjust the values of d_i and b_{ij} so that, for all $x(0)$, we minimise $x_i(\infty)$ for all i (or some i)

- Decentralised control: each node i uses $x_i(t)$ to implement feedback control [1,2,3]
  - Does not require knowledge of node parameters d_i, b_{ij}
  - Dynamic updating of control input, i.e. “closing the loop”

[1] Y. Wang, S. Gracy, C. A. Uribe, H. Ishii, and K. H. Johansson. A State Feedback Controller for Mitigation of Continuous-Time Networked SIS Epidemics, 2022

[2] J. Liu, P. E. Pare, A. Nedic, C. Y. Tang, C. L. Beck, and T. Basar. Analysis and control of a continuous-time bi-virus model. *IEEE Transactions on Automatic Control*, 2019.

[3] M. Ye, J. Liu, B. D. O. Anderson, and M. Cao. Applications of the Poincare-Hopf Theorem: Epidemic Models and Lotka-Volterra Systems. *IEEE Transactions on Automatic Control*, 2022.

A class of decentralised feedback controllers

$$\dot{x}_i(t) = -d_i x_i(t) + (1 - x_i(t)) \sum_{j \in \mathcal{N}_i} b_{ij} x_j(t)$$

Consider a class of feedback controllers of the form

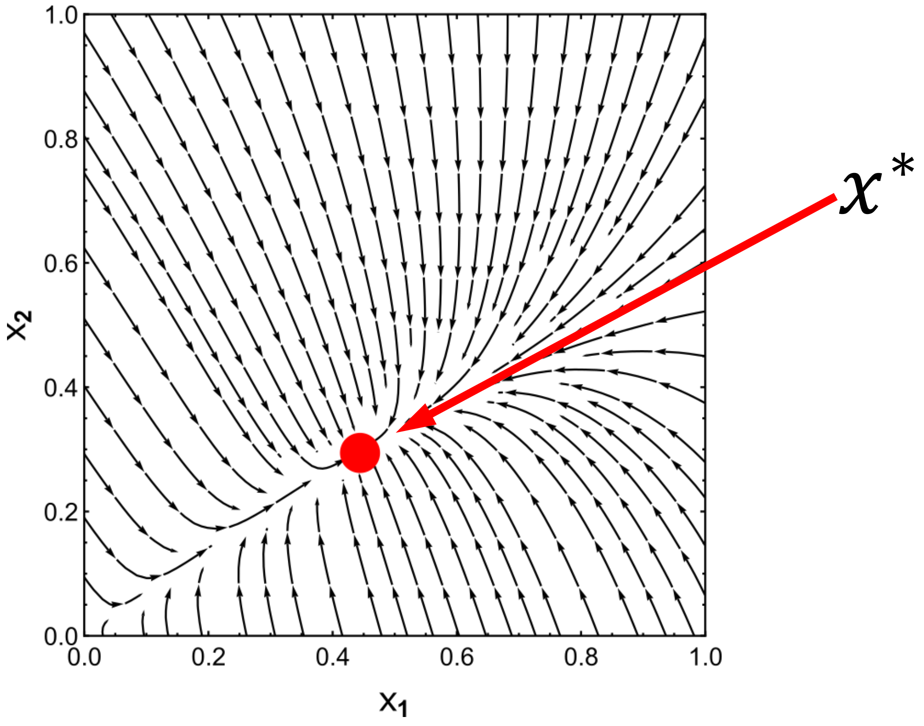
$$u(x_i(t)) = d_i + h_i(x_i(t))$$

- $d_i > 0$ is the base recovery rate of population i
- $h_i(x_i): [0,1] \to \mathbb{R}_{\geq 0}$ is smooth, monotonically nondecreasing and satisfies $h_i(0) = 0$
- E.g. $h_i(x_i) = k_i x_i$ for some control gain $k_i > 0$

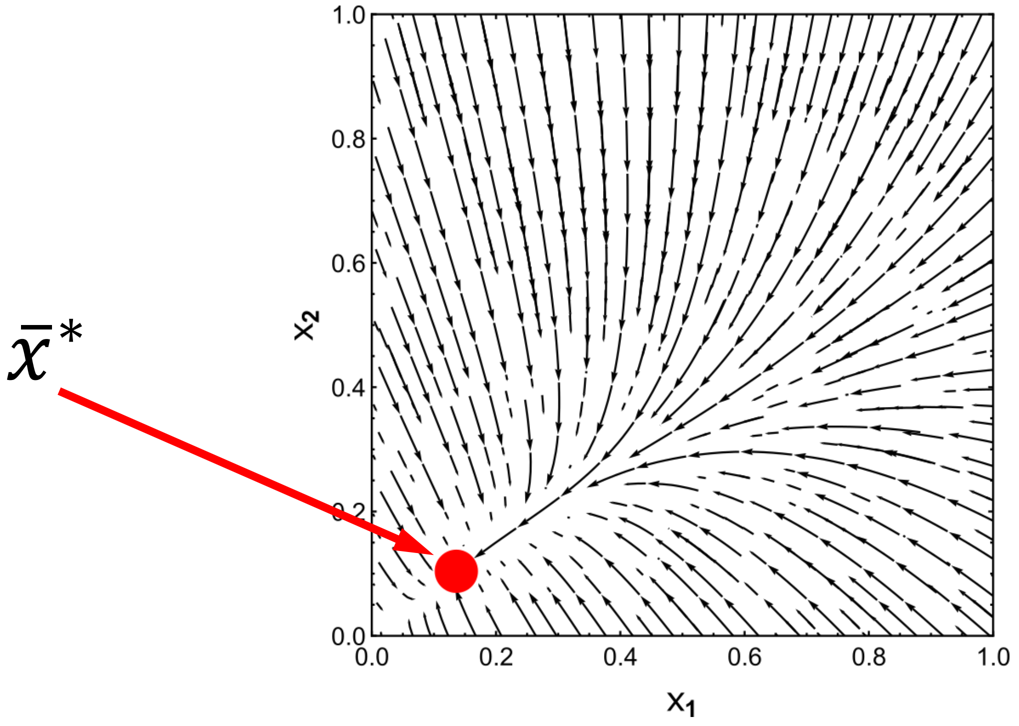
$$\dot{x}_i(t) = -u_i(x_i(t)) x_i(t) + (1 - x_i(t)) \sum_{j \in \mathcal{N}_i} b_{ij} x_j(t)$$

Cutting a long-story short…

Impossibility result: The decentralised controllers cannot eradicate the disease if the disease is endemic in the uncontrolled network!



Uncontrolled network $R_0 > 1$



Controlled network, $h_1 = 0.5x_1^{0.5}$, $h_2 = 0.9x_2$

Reflections

$$u(x_i(t)) = d_i + h_i(x_i(t))$$

- $h_i(x_i)$: $[0,1] \to \mathbb{R}_{\geq 0}$ is smooth, monotonically nondecreasing and satisfies $h_i(0) = 0$
- E.g. $h_i(x_i) = k_i x_i$ for some control gain $k_i > 0$

$$\dot{x}_i(t) = -u_i(x_i(t))x_i(t) + (1 - x_i(t)) \sum_{j \in \mathcal{N}_i} b_{ij} x_j(t)$$

- Proof of impossibility result
  - Poincare-Hopf Theorem to establish existence of endemic equilibrium for controlled system
  - Monotone dynamical systems theory to establish global exponential convergence

- Can we design a more creative control algorithm to eliminate the disease?

- Introduction

- Epidemic spreading on a network
  - Full network control
  - Partial network control

- Conclusions and future work

[1] L. Walsh, **M. Ye**, B. D.O. Anderson, and Z. Sun. Decentralised adaptive-gain control for eliminating epidemic spreading on networks. Submitted journal paper. ArXiv: https://arxiv.org/abs/2305.16658

[2] L. Walsh, **M. Ye**, B. D.O. Anderson, and Z. Sun. Decentralised adaptive-gain control for the Susceptible--Infected--Susceptible network epidemic model. 22nd *IFAC World Congress*, Yokohama, Japan, 2023

Decentralised adaptive-gain control (infection rate)

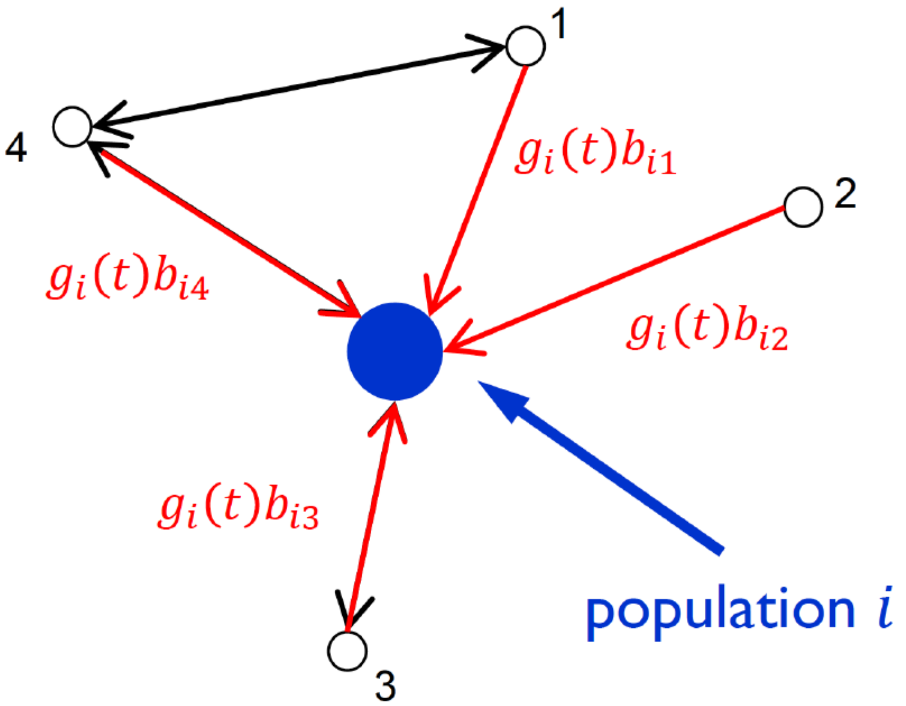
$$\dot{x}_i(t) = -d_i x_i(t) + \big(1 - x_i(t)\big) g_i(t) \sum_{j \in N_i} b_{ij}\, x_j(t)$$

$$\dot{g}_i(t) = -\phi_i\big(x_i(t)\big) g_i(t), \qquad g_i(0) = 1$$

- $\phi_i(x_i) = \alpha_i x_i^p$ is the adaptation function, where $p \in \mathbb{N}_+$ is common to all nodes and $\alpha_i > 0$ is a tuning parameter

- $g_i(t) \in [0,1]$ for all t, i.e. well-defined

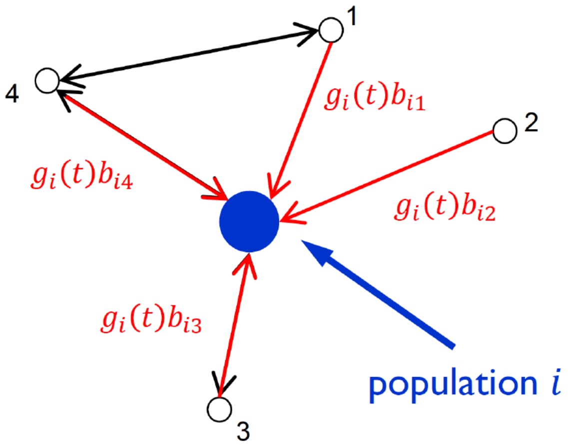
- Gain $g_i(t)$ represents NPIs that reduces mobility (and hence infection rate) to entirety of population i



Network dynamics

$$\begin{bmatrix} \dot{x}(t) \\ \dot{g}(t) \end{bmatrix} = \begin{bmatrix} -D + \big(I_n - X(t)\big)G(t)B \\ AX(t)^p \end{bmatrix} \begin{bmatrix} x(t) \\ g(t) \end{bmatrix}$$

- $x = [x_1, \dots, x_n]^\top$, $g = [g_1, \dots, g_n(t)]$, $D = \mathrm{diag}(d_i)$ and $X(t) = \mathrm{diag}(x_i(t))$, $G = \mathrm{diag}(g_i(t))$, $A = \mathrm{diag}(\alpha_i)$ and $B = \{b_{ij}\}$



- Intuitively, with $g_i(t)$ monotonically decreasing, we can easily prove $x_i \to 0$ as $t \to \infty$

- **Main challenge:** prove that $\lim_{t\to\infty} g_i(t) = \bar{g}_i > 0$ for all i, i.e. we avoid having to totally lock down any one population

- It is easy to prove there exists i such that $\bar{g}_i > 0$

Main result for infection rate control

$$\begin{bmatrix} \dot{x}(t) \\ \dot{g}(t) \end{bmatrix} = \begin{bmatrix} -D + \big(I_n - X(t)\big)G(t)B & \\ AX(t)^p & \end{bmatrix} \begin{bmatrix} x(t) \\ g(t) \end{bmatrix}$$

Theorem: Consider the system above, with B irreducible, A and D positive diagonal, and $\rho(D^{-1}B) > 1$. Then for all $x(0) \in [0,1]^n$ there holds

- $\lim_{t\to\infty} x(t) = \mathbf{0}_n$ and $\lim_{t\to\infty} g(t) = \bar{g} > \mathbf{0}_n$

- Brief sketch of proof:

$$g_i(t) = g_i(0)e^{-\int_0^t \phi_i(x_i(s))ds}$$

  - Assume some gains $g_i(t) \to \bar{g}_i > 0$ while other gains $g_j(t) \to 0$
  - For the $g_i(t) \to \bar{g}_i$, standard application of Barbalat's lemma establishes $x_i(t) \to 0$
  - For the $g_j(t) \to 0$, we show by contradiction that no such j can exist
  - Two key tools: L^p function spaces, and vector differential inequalities [1], the latter being applicable for monotone systems

[1] W. Walter, "Ordinary Differential Inequalities in Ordered Banach Spaces," *Journal of Differential Equations*, 1971.

Performance analysis

To help us examine the performance, let us define the controlled reproduction number $R_t = \rho(D^{-1}G(t)B)$

Proposition:
- R_t is monotonically decreasing in t, and $\lim_{t\to\infty} R_t = R_\infty \leq 1$.
- If $p = 1$, then $R_\infty < 1$ and convergence is exponentially fast.

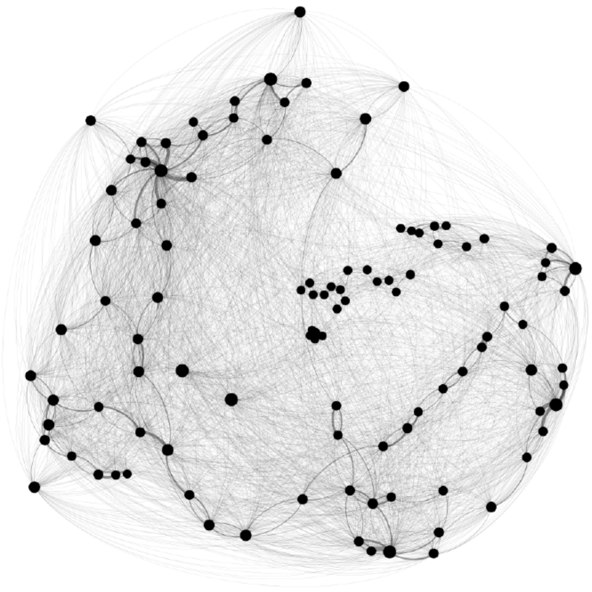
Recall:
$\dot{g}_i(t) = -\alpha_i x_i^p(t) g_i(t)$

- Simulations suggest when $p \geq 2$, convergence can be (but is not always) as slow as $\frac{1}{t}$
- Another auxiliary result: $\lim_{t\to\infty} g_i(t) \leq e^{-\frac{\alpha_i x_i^p(0)}{p d_i}}$

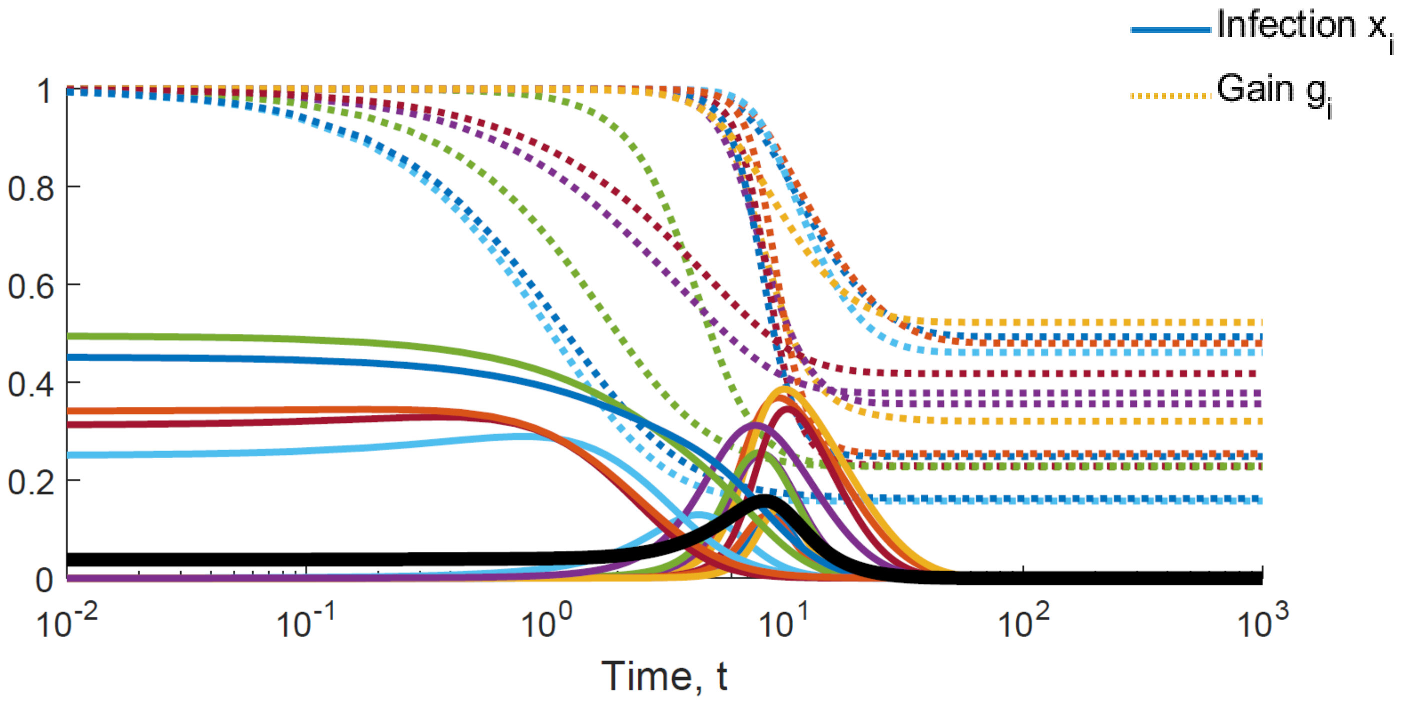
Simulation example

A real-world network structure capturing people mobility patterns between 107 Italian provinces

Uncontrolled network $R_0 > 1$
Controller $\dot{g}_i = \alpha_i x_i$



(a) Italy Network



(b) Network dynamics

Partial network control (infection rate)

Previously we assumed every node executed its adaptive controller in a decentralised manner

$$\dot{x}_i(t) = -d_i x_i(t) + \big(1 - x_i(t)\big) g_i(t) \sum_{j \in N_i} b_{ij}\, x_j(t)$$

$$\dot{g}_i(t) = -\phi_i\big(x_i(t)\big) g_i(t), \qquad g_i(0) = 1, \qquad \phi_i = \alpha_i x_i^p$$

A natural follow up question is: can we achieve the same (or similar) results by controlling a subset of the nodes?

Partial network control

We define $C = \{i \in \{1,2,\dots,n\} \mid \alpha_i > 0\}$ and $U = \{i \in \{1,2,\dots,n\} \mid \alpha_i = 0\}$ as the set of controlled and uncontrolled nodes

$$\dot{x}_i(t) = -d_i x_i(t) + \big(1 - x_i(t)\big) g_i(t) \sum\nolimits_{j \in N_i} b_{ij}\, x_j(t)$$

$$\dot{g}_i(t) = -\phi_i\big(x_i(t)\big) g_i(t), \qquad g_i(0) = 1, \qquad \phi_i = \alpha_i x_i^p$$

Key problems to consider when given $D = \mathrm{diag}(d_i)$ and $B = \{b_{ij}\}$

- Does there exist a pair (C, U), with $C \cup U = \{1,2,\dots,n\}$ such that $x_i(t) \to 0$ for all i and $g_k(t) \to \bar{g}_k > 0$ for all $k \in C$?

- If one or more pairs (C, U) exist, can we propose an iterative algorithm that selects a suitable pair (C, U)

Existence of a pair (C, U)

Without loss of generality, let $U = \{1,2, \ldots, k\}$ and $C = \{k+1, k+2, \ldots n\}$ as the set of uncontrolled and controlled nodes (we can always reorder the nodes)

Partition D and B as

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \quad D = \begin{bmatrix} D_1 & \mathbf{0}_{k\times(n-k)} \\ \mathbf{0}_{(n-k)\times k} & D_2 \end{bmatrix}$$

Theorem: The following two statements are equivalent

- For all $x(0) \in [0,1]^n$ there holds $\lim_{t\to\infty} x(t) = \mathbf{0}_n$ and $\lim_{t\to\infty} g_i(t) = \bar{g}_i > 0$ for all $i \in C$
- The matrix $-D_1 + B_{11}$ is Hurwitz, or $\rho\left(D_1^{-1}B_{11}\right) < 1$

- Proof employs heavy use of M-matrices, which are a special class of matrices often appearing in network systems (Laplacian matrix is an M-matrix), along with Centre Manifold Theory, and (again) differential inequalities

Existence of a pair (C, U)

Partition D and B as

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \qquad D = \begin{bmatrix} D_1 & \mathbf{0}_{k\times(n-k)} \\ \mathbf{0}_{(n-k)\times k} & D_2 \end{bmatrix}$$

Theorem: The following two statements are equivalent

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- The matrix $-D_1 + B_{11}$ is Hurwitz, or $\rho\left(D_1^{-1} B_{11}\right) < 1$

- Existence of a pair (C, U) is thus equivalent to existence of a reordering of nodes such that $-D_1 + B_{11}$ is Hurwitz

- Intuitively: the uncontrolled subnetwork must be able to eradicate the disease itself

- Any node i such that $d_i \leq b_{ii}$ must belong in C

Iterative algorithm for finding (C, U)

- Our iterative algorithm relies heavily on the result of Duan et al. 2022 [1], including the idea of a "sum-cycle gain"

- It requires identifying all simple cycles in a network (computationally intensive)

1. Begin by assuming all nodes are uncontrolled
2. Place all nodes with $d_i \leq b_{ii}$ into the control set C
3. In the graph of U, iterate as follows
   - Select one cycle in U, and place one of its nodes into C
   - Check the "sum-cycle gain condition"

Key result: Algorithm always terminates with C and U both non-empty, assuming the existence condition was met

[1] X. Duan, S. Jafarpour, and F. Bullo, "Graph-theoretic stability conditions for Metzler matrices and monotone systems," *SIAM Journal on Control and Optimization*, 2021.

Iterative algorithm for finding (C, U)

1. Begin by assuming all nodes are uncontrolled
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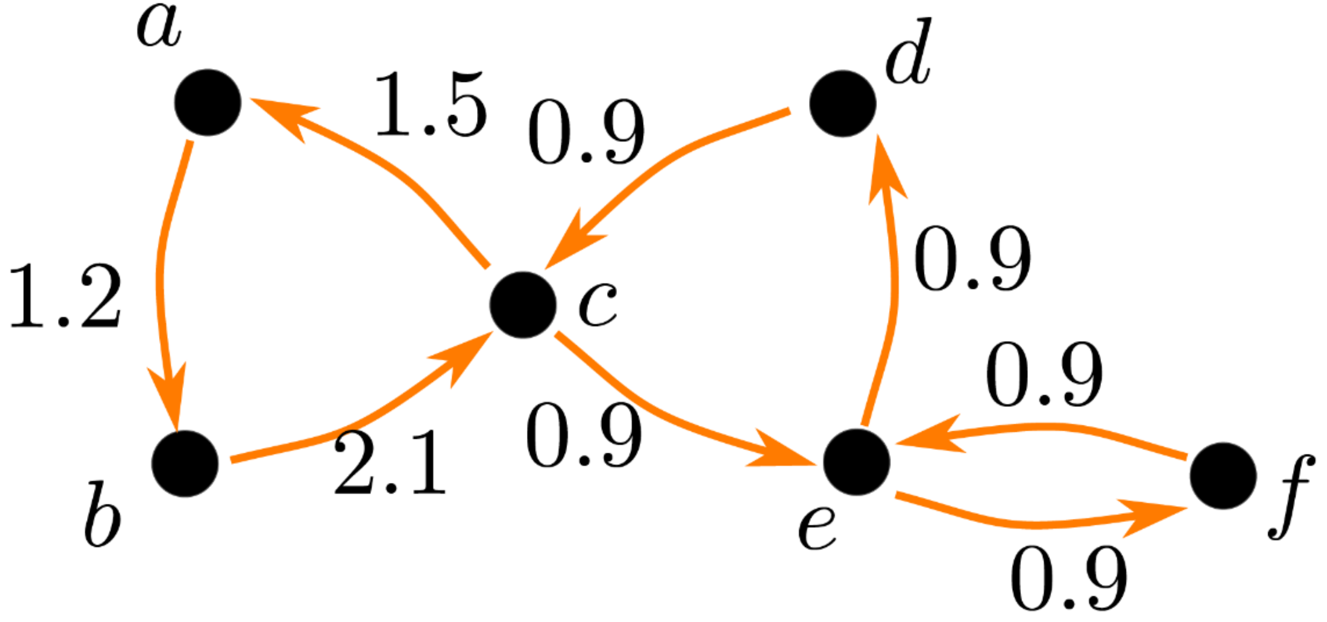
3. In the graph of U, iterate as follows
   - Select one cycle in U, and place one of its nodes into C
   - Check the "sum-cycle gain condition"

$d_i = 2$ for all nodes

$b_{ii} = 1$ for all nodes except
$b_{aa} = 4$



Control of recovery rates

We focused on controlling the infection rate (e.g. via NPIs)

$$\dot{x}_i(t) = -d_i x_i(t) + \big(1 - x_i(t)\big) g_i(t) \sum_{j \in N_i} b_{ij}\, x_j(t)$$

$$\dot{g}_i(t) = -\phi_i\big(x_i(t)\big) g_i(t), \qquad g_i(0) = 1$$

But we can easily consider control of recovery rates (medical interventions)

$$\dot{x}_i(t) = -d_i g_i(t) x_i(t) + \big(1 - x_i(t)\big) \sum_{j \in N_i} b_{ij}\, x_j(t)$$

$$\dot{g}_i(t) = \phi_i\big(x_i(t)\big), \qquad g_i(0) = 1$$

$$\phi_i = \alpha_i x_i^p$$

Mutatis mutandis, all results presented earlier are the same

Conclusions

- Proposed a decentralised adaptive-gain control algorithm for each node in an SIS network model, for both control of infection rate and recovery rate

- Theoretical results establish that the controller can drive the infections at every node to zero, while the gains converge to positive values

- Exponential convergence for a subset of the control parameters, and what appears to be $1/t$ convergence rate for another subset

- Considered the situation where only a subset of the nodes can be controlled
  - Established a necessary and sufficient condition for existence of such a subset
  - Proposed an iterative algorithm to select a suitable control node set

- Key tools used: L^p stability, M-matrix theory, vector differential inequalities

Current and future work

- Improved computational efficiency of the iterative selection algorithm
- Consider a mixture of recovery and infection rate control in the same network
- Consider adaptive edge-based control, so that each node controls incoming edges independently
- More sophisticated gain design so that $g_i(t)$ is not monotonic
- More realistic implementation by updating $g_i(t)$ in a piece-wise manner
  - More reflective of phased introduction of interventions in the real world
  - Periodic updating of $g_i(t)$
  - Event-triggered updating, with intelligent selection of triggering function to balance frequency of update and timely removal of disease

Thanks! Any Questions?

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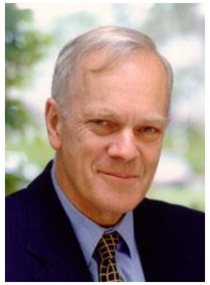

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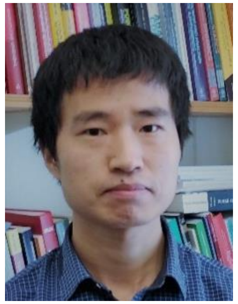

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GOVERNMENT OF WESTERN AUSTRALIA

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- Introduction

- Part I: Epidemic spreading on a network

- Part II: Population games

[1] L. Zino, M. Ye, A. Rizzo, G. C. Calafiore. On Adaptive-Gain Control of Replicator Dynamics in Population Games. To appear in *IEEE Conference on Decision and Control*, Singapore, 2023. ArXiv: https://arxiv.org/abs/2306.14469

Evolutionary game theory

- Population of individuals repeatedly engaging in strategic interactions with associated payoffs [1]

- Models a diverse range of decision-making and evolutionary processes

- **Objective:** steer the population towards a desired state, e.g. consensus adoption of sustainable practices

[1] W. H. Sandholm, *Population Games and Evolutionary Dynamics*. Cambridge University Press, 2010.

Two-player matrix game

- Two players play against each other, choose between two mutually exclusive actions: 1 and 2
- Player i, playing against player j, receives a payoff determined by payoff matrix $\boldsymbol{A}$:

$$\begin{array}{cc} & \begin{array}{cc} x_j = 1 & x_j = 2 \end{array} \\ \begin{array}{c} x_i = 1 \\ x_i = 2 \end{array} & \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{array}$$

- Three classes of games based on payoff matrix entry values [1]
  - **Coordination game:** $a > c$ and $d > b$. Adopting the same strategy (consensus) provides greatest payoff (social conventions and norms)
  - **Anti-coordination game:** $c > a$ and $b > d$. Adopting opposite strategy (disagreement) provides greatest payoff (traffic congestion, queueing)
  - **Dominant-strategy game:** $c > a$ and $d > b$ (or $a > c$ and $b > d$). One particular strategy provides greatest payoff irrespective of opponent choice (Prisoner's dilemma)

[1] J. Riehl, P. Ramazi, and M. Cao. A survey on the analysis and control of evolutionary matrix games. *Annual Reviews in Control*, 2028

Population games and replicator equation

- Population of individuals where each individual plays two-player game against all others, and individuals revise their strategy using the replicator equation [1]

- Let $x(t) \in [0,1]$ denote the fraction of the population adopting action 1. Then:

$$\dot{x}(t) = x(1-x)((a+d-b-c)x + b - d)$$

- Equilibria properties:
  - $x = 0$ and $x = 1$ are both equilibria (pure strategy NE), also called consensus states
  - $x^* = \frac{d-b}{a+d-b-c}$ is an equilibrium for coordination/anti-coordination game (mixed strategy NE)

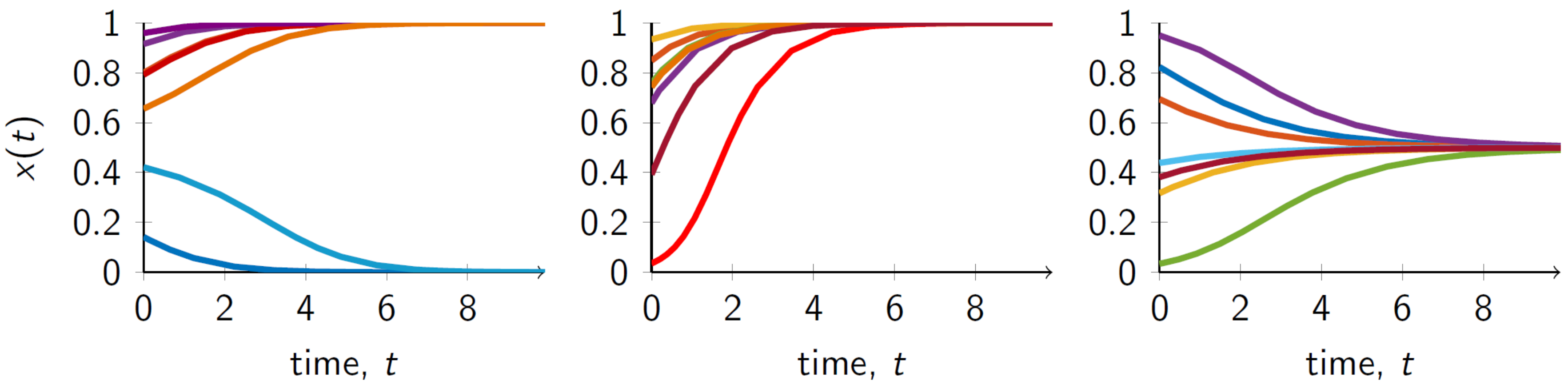
[1] W. H. Sandholm, *Population Games and Evolutionary Dynamics.* Cambridge University Press, 2010.

Asymptotic replicator dynamics

- Let $x(t) \in [0,1]$ denote the fraction of the population adopting action 1. Then:

$$\dot{x}(t) = x(t)(1 - x(t))((a + d - b - c)x(t) + b - d)$$

Theorem: Consider the replicator dynamics above. Then, the following hold:
- For a coordination game, $x(t) \to 0$ if $x(0) < x^*$ and $x(t) \to 1$ if $x(0) > x^*$
- For an anti-coordination game, $x(t) \to x^*$ for all $x(0) \in (0,1)$
- For a dominant-strategy game with $c > a$ and $d > b$, $x(t) \to 0$ for all $x(0) < 1$



[1] W. H. Sandholm, *Population Games and Evolutionary Dynamics*. Cambridge University Press, 2010.

Controlling the replicator dynamics

$$\dot{x}(t) = x(t)(1 - x(t))((a + d - b - c)x(t) + b - d)$$

- We wish to steer the replicator dynamics to a desired equilibrium $\bar{x}$ (setpoint regulation)
- Promote cooperation in social dilemmas, adoption of sustainable innovations, etc.

- Existing methods
  - Directly control the actions of some individuals (not always feasible) [1,2]
  - Open-loop control with permanent instantaneous change to payoff matrix (requires knowledge of game and unnecessarily costly in the long-term) [3]

- Adaptive-gain approach: closed-loop control with limited information on the game

[1] M. Ye, L. Zino, Ž. Mlakar, J. W. Bolderdijk, H. Risselada, B. M. Fennis, and M. Cao. Collective patterns of social diffusion are shaped by individual inertia and trend-seeking. *Nature Communications*, 2021.

[2] D. Centola, J. Becker, D. Brackbill, and A. Baronchelli, “Experimental evidence for tipping points in social convention,” *Science*, 2018.

[3] J. Riehl, P. Ramazi, and M. Cao, “Incentive-Based Control of Asynchronous Best-Response Dynamics on Binary Decision Networks,” *IEEE Transactions on Control of Networked Systems*, 2018.

Problem formulation

$$A(t) = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\text{payoff matrix}} + \underbrace{\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}}_{\text{control matrix}} \underbrace{g(t)}_{\text{gain}}$$

- Objective: steer the replicator dynamics to a desired equilibrium $\bar{x}$ (setpoint regulation)

- Gain adaptively changes via adaptation function $\phi(x) : [0,1] \rightarrow \mathbb{R}$ via $\dot{g}(t) = \phi(x)g(t)$

- Design a pair (G, ϕ) such that i) $x(t) \rightarrow \bar{x}$ for all $x(0) \in (0,1)$, and ii) $g(t) \rightarrow \bar{g}$

- **Problem 1:** $\bar{x}$ is a locally (but not globally) stable equilibrium (coordination games)
- **Problem 2:** $\bar{x}$ is an unstable consensus equilibrium (anti-coordination or dominant-strategy)
- **Problem 3:** $\bar{x}$ is an arbitrary point, and not an equilibrium of any game

Problem formulation

$$A(t) = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\text{payoff matrix}} + \underbrace{\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}}_{\text{control matrix}} \underbrace{g(t)}_{\text{gain}}$$

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- Design a pair (G, ϕ) such that i) $x(t) \to \bar{x}$ for all $x(0) \in (0,1)$, and ii) $g(t) \to \bar{g}$

- **Problem 1:** $\bar{x}$ is a locally (but not globally) stable equilibrium (coordination games)
  - Two locally stable equilibria $x = 0$ and $x = 1$, and a saddle point $x^* = \frac{d-b}{a+d-b-c} \in (0,1)$ splitting the basins of attraction
  - Without loss of generality, set $\bar{x} = 0$, i.e. we want to reach a consensus on action 2
  - Controller needs to drive $x(t) \to 0$ for all $x(0) \in [x^*, 1)$

Problem 1: Innovation gain

- $G_{21} = 1,\ G_{11} = G_{12} = G_{22} = 0$

$$A(t) = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\text{payoff matrix}} + \underbrace{\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}}_{\text{control matrix}} \underbrace{g(t)}_{\text{gain}}$$

$$\dot{x}(t) = x(1-x)\big((a+d-b-c)x + b - d + gx\big)$$
$$\dot{g} = \phi(x)g, \qquad g(0) > 0$$

Theorem: The innovation gain controller solves Problem 1 if:
- $\phi(x) < 0$ for $x \in [0, \delta]$ where δ is such that $x \in [0, \delta]$ is in the basin of attraction of $x = 0$;
- $\phi(x) > 0$ for $x \in (\delta, 1]$.

- Key takeaway: we need only an estimate of the basin of attraction of $x = 0$, and we only need to increase interventions when outside this estimated basin of attraction

- Example function: $\phi(x) = k(x - h)$ where $h = \delta$, and k is a tuning parameter

Problem 1: Coordination gain

- $G_{22} = 1,\ G_{11} = G_{12} = G_{21} = 0$

$$A(t) = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\text{payoff matrix}} + \underbrace{\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}}_{\text{control matrix}} \underbrace{g(t)}_{\text{gain}}$$

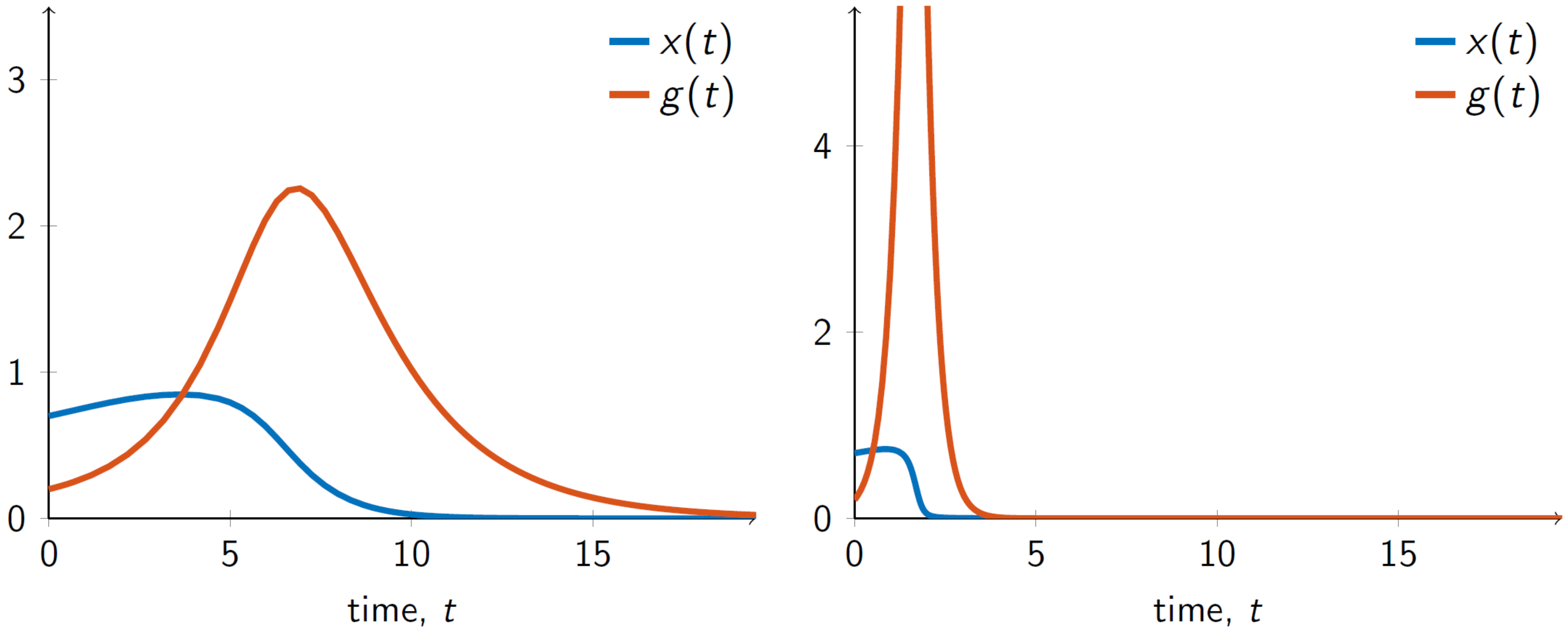
$$\dot{x}(t) = x(1-x)\big((a+d-b-c)x + b - d - g(1-x)\big)$$
$$\dot{g} = \phi(x)g, \qquad g(0) > 0$$

Theorem: The coordination gain controller solves Problem 1 if:

- $\phi(x) < 0$ for $x \in [0,\delta]$ where $\delta > 0$ is such that $x \in [0,\delta]$ is in the basin of attraction of $x = 0$;
- $\phi(x) > 0$ for $x \in (\delta, 1]$;
- $\phi(x) > a - c$ for all $x \in [1-\epsilon, 1]$, for some $\epsilon > 0$

- Sketch of proof: when $g(t)$ is sufficiently large, $x = 1$ is a repeller. The third condition ensures $g(t)$ grows fast enough as $x(t)$ approaches 1, so that it can never reach 1

Simulation example



- Left: innovation gain controller; Right: coordination gain controller
- Coordination gain can provide faster convergence, but higher peak gain and requires more information

Problems 2 and 3

Problem 2: $\bar{x}$ is an unstable consensus equilibrium (anti-coordination or dominant-strategy)

Problem 3: $\bar{x}$ is an arbitrary point, and not an equilibrium of any game

Can be dealt with using similar adaptive controllers. Some key findings:

- It is impossible to solve Problem 2 using the innovation gain approach.
- Using coordination gain approach solves Problem 2, with very mild and general conditions on ϕ. However, $g(t)$ is monotonically increasing, and $g(t) \to \bar{g} < \infty$
- The innovation gain approach solves Problem 3, but the form of ϕ is much more restrictive

Conclusions

- Formulated three setpoint regulation problems for the replicator dynamics using adaptive-gain control

- For Problem 1, both innovation gain and coordination gain solves the problem, with intuitive tradeoffs between the two approaches

- Problems 2 and 3 can be similarly solved

- The adaptive-gain controllers require little information about the gain (only estimates are needed, and these can be as conservative as one wishes)

Current and future work

- Control of networked population games (similar to the SIS network model)

- Optimised design of ϕ that balances: minimisation of peak gain value, maximization of convergence speed, and minimisation of total control effort $\int g(s).ds$

- More sophisticated gain design so that $g(t)$ is not monotonic for Problem 2

- More realistic implementation by updating $g(t)$ in a piece-wise manner
  - More reflective of phased introduction of interventions in the real world
  - Periodic updating of $g(t)$
  - Event-triggered updating, with intelligent selection of triggering function to balance frequency of update and timely removal of disease